

BISPECTRALITY OF MULTIVARIABLE RACAH-WILSON POLYNOMIALS

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ABSTRACT. We construct a commutative algebra \mathcal{A}_x of difference operators in \mathbb{R}^p , depending on $p + 3$ parameters which is diagonalized by the multivariable Racah polynomials $R_p(n; x)$ considered by Tratnik [27]. It is shown that for specific values of the variables $x = (x_1, x_2, \dots, x_p)$ there is a hidden duality between n and x . Analytic continuation allows us to construct another commutative algebra \mathcal{A}_n in the variables $n = (n_1, n_2, \dots, n_p)$ which is also diagonalized by $R_p(n; x)$. Thus $R_p(n; x)$ solve a multivariable discrete bispectral problem in the sense of Duistermaat and Grünbaum [8]. Since a change of the variables and the parameters in the Racah polynomials gives the multivariable Wilson polynomials [26], this change of variables and parameters in \mathcal{A}_x and \mathcal{A}_n leads to bispectral commutative algebras for the multivariable Wilson polynomials.

CONTENTS

1. Introduction	2
2. Triangular difference operator in \mathbb{R}^p	4
2.1. Basic notations and definitions	4
2.2. The operator \mathcal{L}_p	5
3. Multivariable Racah polynomials	8
3.1. The operator \mathcal{L}_p in terms of the shift operators	8
3.2. Racah inner product and polynomials	11
3.3. Self-adjointness of \mathcal{L}_p	12
3.4. Admissibility of \mathcal{L}_p and the commutative algebra \mathcal{A}_x	14
4. Bispectrality	17
4.1. Duality	17
4.2. The dual algebra \mathcal{A}_n	20
5. Examples of other bispectral families of orthogonal polynomials	22
5.1. Wilson polynomials	23
5.2. Hahn polynomials	23
5.3. Jacobi polynomials	26
5.4. Krawtchouk and Meixner polynomials	28
Appendix A. Explicit formulas in dimension two	30
A.1. Racah polynomials	30
A.2. Jacobi polynomials	32

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A.3. Krawtchouk polynomials	33
Acknowledgments	34
References	34

1. INTRODUCTION

It is well known that any family of polynomials orthogonal with respect to a positive measure supported on the real line satisfies a three term recurrence relation,

$$d_1(n)p_{n+1}(x) + d_0(n)p_n(x) + d_{-1}(n)p_{n-1}(x) = xp_n(x). \quad (1.1)$$

Furthermore it was shown by Bochner [5] that the classical orthogonal polynomials of Jacobi, Hermite and Laguerre can be characterized by the fact that they satisfy a second order differential equation of the form

$$c_2(x)\frac{d^2}{dx^2}p_n(x) + c_1(x)\frac{d}{dx}p_n(x) + c_0(x)p_n(x) = \mu(n)p_n(x). \quad (1.2)$$

Thus the classical orthogonal polynomials are simultaneously eigenfunctions of a difference operator in n and a differential operator in x , i.e. they are part of a discrete-continuous version of the bispectral problem. The bispectral problem is concerned with finding differential or difference operators that have common eigenfunctions. More precisely, find a differential or a difference operator L_n , independent of x , acting on functions of n , and a differential or a difference operator L_x which is independent of n acting on functions of x , so that

$$\begin{aligned} L_n\psi(x, n) &= f(x)\psi(x, n) \\ L_x\psi(x, n) &= \mu(n)\psi(x, n), \end{aligned} \quad (1.3)$$

where $f(x)$ and $\mu(n)$ are functions of x and n respectively. This problem has appeared in many areas of mathematics, physics and engineering such as limited angle tomography, soliton equations and their master symmetries, particle systems, algebraic geometry and representations of infinite dimensional Lie algebras, see for instance [8, 10, 31, 29, 4, 12] as well as the papers in [14].

While in (1.1) and (1.2) L_n is a difference operator and L_x is a differential operator the bispectral problem considered in Duistermaat and Grünbaum [8] was for both L_n and L_x to be differential operators which give rise to the continuous-continuous part of the bispectral problem. An advantage of this is that it puts both variables on an equal footing so it may be possible to obtain L_n from L_x or vice versa via some hidden symmetry. In order to pursue this strategy for orthogonal polynomials we look for more general families of polynomials which have the classical polynomials as limiting cases and for which L_n and L_x are difference operators. Indeed, for the discrete classical orthogonal polynomials (Charlier, Meixner, Krawtchouk and Hahn) equation (1.2) is replaced by a difference equation of the form

$$C_1(x)p_n(x+1) + C_0(x)p_n(x) + C_{-1}(x)p_n(x-1) = \mu(n)p_n(x), \quad (1.4)$$

see for instance [23], while (1.1) is the same. The symmetry between n and x , which allows us to connect the two equations, is transparent in the cases of the Charlier, Meixner and Krawtchouk polynomials from their explicit formulas in terms of hypergeometric series. However, this is no longer true for the Hahn polynomials. One essential difference between equations (1.1) and (1.4) for Hahn polynomials is that

the eigenvalue $\mu(n)$ is quadratic in n , while the eigenvalue in (1.1) is linear. Also the coefficients of the difference operator (1.4) are polynomials of x , while the coefficients in (1.1) are rational functions of n . Thus, we naturally arrive at the Racah polynomials, where the eigenvalue on the right-hand side of (1.1) is also quadratic, and the symmetry between n and x can be seen from the ${}_4F_3$ representation. A similar duality holds also for the most general family - the Askey-Wilson polynomials [2]. In fact, the duality is even deeper and it can be extended beyond the polynomial case, to bi-infinite difference operators [13].

One beautiful extension of the above theory to orthogonal polynomials of more than one variable is related to the theory of symmetric functions known as Macdonald-Koornwinder polynomials, see for instance [15, 20, 21, 28]. A deeper understanding of the duality stems from affine Hecke algebras considerations, introduced by Cherednik [7] in his proof of some conjectures about Macdonald polynomials (see also [24] for non-reduced root systems BC_n).

In a different vein, one can also look for multivariable orthogonal polynomials satisfying an equation similar to (1.4). The first obstacle is the fact that in the case of more than one variable we have more than one monomial of total degree n , for $n > 0$. A natural classification of the possible weights should be independent of the way we order the monomials of a given degree when we apply the Gram-Schmidt process. This led to the notion of admissible difference operators, which was used by Iliev and Xu [16] to characterize discrete multivariable orthogonal polynomials, satisfying (partial) second-order difference equations analogous to (1.4). It is interesting to notice that the multivariable versions of Hahn, Krawtchouk and Meixner polynomials considered there had already appeared in the literature in different applications [18, 22]. An intriguing question left open in [16] is whether these polynomials are bispectral and if the bispectrality can be derived from duality. Notice that this time a difference equation similar to (1.1) is no longer an automatic consequence of the orthogonality, and its existence will depend on the way we order the monomials of total degree n . By analogy with the one dimensional theory if we hope to get the bispectrality from duality we need to go at least to multivariable Racah polynomials.

In the present paper we answer affirmatively the questions raised above by considering the multivariable Racah polynomials defined by Tratnik [27]. These polynomials are orthogonal in \mathbb{R}^p with respect to a weight depending on $p+2$ real parameters $\beta_0, \beta_1, \dots, \beta_{p+1}$ and a positive integer N . First, we define a difference operator $\mathcal{L}_p(x; \beta; N)$ in the variables x_1, x_2, \dots, x_p which is triangular (see Definition 2.3) and self-adjoint with respect to the multivariable Racah weight. Here we have put $\beta = (\beta_0, \beta_1, \dots, \beta_{p+1}) \in \mathbb{R}^{p+2}$ and $x = (x_1, x_2, \dots, x_p) \in \mathbb{R}^p$. As an immediate corollary we see that any set of orthogonal polynomials with respect to Racah inner product will be eigenfunctions of $\mathcal{L}_p(x; \beta; N)$.

It is a bit surprising that this difference operator has a much more complicated structure than the operator for Hahn polynomials. Besides the fact that its coefficients are rational functions of x , it is supported on the cube $\{-1, 0, 1\}^p$. This means that it has 3^p rational coefficients, compared to $p(p+1)$ polynomial coefficients for the operator corresponding to multivariable Hahn polynomials. In order to define and study the basic properties of this operator, we introduce p involutions on the associative algebra of difference operators in x with rational coefficients, see Section 2 for more details. Similar involutions (on the dual n -side) played a crucial

role in [11, 13] in the construction of orthogonal polynomials satisfying higher-order differential or q -difference equations, see Remark 4.8 for more details.

Next we focus on the basis of orthogonal polynomials $\{R_p(n; x; \beta; N) : n \in \mathbb{N}_0^p \text{ such that } |n| = n_1 + \dots + n_p \leq N\}$ constructed by Tratnik. We define a commutative algebra \mathcal{A}_x generated by p difference operators in the variables x_1, x_2, \dots, x_p , which is diagonalized by $\{R_p(n; x; \beta; N)\}$. An analytic continuation argument shows that these equations are valid even when $N \notin \mathbb{N}_0$, in which case the polynomials $\{R_p(n; x; \beta; N)\}$ are defined for all $n \in \mathbb{N}_0^p$. All this is the content of Section 3.

In the next section, we prove that appropriately normalized, the polynomials $\{R_p(n; x; \beta; N)\}$ possess a certain duality between the variables n and x . The proof is obtained by applying twice an identity of Whipple connecting the terminating Saalschützian ${}_4F_3$. The duality allows us to define an isomorphic commutative algebra \mathcal{A}_n of difference operators in the variables n_1, n_2, \dots, n_p which is also diagonalized by $\{R_p(n; x; \beta; N)\}$, thus proving the bispectrality.

In Section 5 we illustrate the bispectrality of other families of multivariable orthogonal polynomials. In Subsection 5.1 we show that a change of the variables and the parameters in the Racah polynomials gives the multivariable Wilson polynomials [26]. Therefore, changing the variables and the parameters in \mathcal{A}_x and \mathcal{A}_n leads to bispectral commutative algebras for the multivariable Wilson polynomials. In Subsections 5.2–5.3 we consider limiting cases which lead to bispectral commutative algebras for multivariable Hahn and Jacobi polynomials. The explicit formulas derived there show that although the operators on the x side simplify tremendously when we go to the Hahn or Jacobi polynomials, the operators on the n side are more or less the same. In the case of Jacobi polynomials, \mathcal{A}_x becomes a commutative algebra of differential operators going back to the work of Appell and Kampé de Fériet [1]. In the last subsection we illustrate the bispectrality and the duality of the multivariable Krawtchouk and Meixner polynomials.

We have collected explicit formulas in dimension two in an Appendix. The reader eager to get a feeling about the complexity of the formulas or willing to look at some examples first is referred to the Appendix.

2. TRIANGULAR DIFFERENCE OPERATOR IN \mathbb{R}^p

In this section we introduce a triangular difference operator and we discuss its basic properties.

2.1. Basic notations and definitions. Consider the ring $\mathcal{P} = \mathbb{R}[x_1, x_2, \dots, x_p]$ of polynomials in p independent variables x_1, x_2, \dots, x_p with real coefficients. We denote by $\mathcal{P}_\lambda = \mathbb{R}[\lambda_1, \lambda_2, \dots, \lambda_p]$ the subring of \mathcal{P} consisting of polynomials in the variables $\lambda_1, \lambda_2, \dots, \lambda_p$, where

$$\lambda_i = \lambda_i(x_i) = x_i(x_i + \beta_i) \quad \text{for } i = 1, 2, \dots, p, \tag{2.1}$$

and $\beta_1, \beta_2, \dots, \beta_p$ are parameters.

For $i = 1, 2, \dots, p$ we define an automorphism I_i of \mathcal{P} by

$$I_i(x_i) = -x_i - \beta_i \text{ and } I_i(x_j) = x_j \text{ for } j \neq i. \tag{2.2}$$

Notice that I_i is an involution (i.e. $I_i \circ I_i = \text{Id}$). The importance of these involutions is summarized in the following remark.

Remark 2.1. The involutions I_i preserve \mathcal{P}_λ . Conversely, if a polynomial $q \in \mathcal{P}$ is preserved by the involutions I_i for $i = 1, 2, \dots, p$ then $q \in \mathcal{P}_\lambda$.

Let $\{e_1, e_2, \dots, e_p\}$ be the standard basis for \mathbb{R}^p . We denote by E_{x_i} , Δ_{x_i} and ∇_{x_i} , respectively, the customary shift, forward and backward difference operators acting on functions f of $x = (x_1, x_2, \dots, x_p)$ as follows

$$\begin{aligned} E_{x_i} f(x) &= f(x + e_i) \\ \Delta_{x_i} f(x) &= f(x + e_i) - f(x) = (E_{x_i} - 1)f(x) \\ \nabla_{x_i} f(x) &= f(x) - f(x - e_i) = (1 - E_{x_i}^{-1})f(x). \end{aligned}$$

Throughout the paper we use the standard multi-index notation. For instance, if $\nu = (\nu_1, \nu_2, \dots, \nu_p) \in \mathbb{Z}^p$ then

$$x^\nu = x_1^{\nu_1} x_2^{\nu_2} \cdots x_p^{\nu_p}, \quad E_x^\nu = E_{x_1}^{\nu_1} E_{x_2}^{\nu_2} \cdots E_{x_p}^{\nu_p}$$

and $|\nu| = \nu_1 + \nu_2 + \cdots + \nu_p$.

Let \mathcal{D}_x be the associative algebra of difference operators of the form

$$L = \sum_{\nu \in S} l_\nu(x) E_x^\nu,$$

where S is a finite subset of \mathbb{Z}^d and $l_\nu(x)$ are rational functions of x . Thus, the algebra \mathcal{D}_x is generated by rational functions of x , the shift operators $E_{x_1}, E_{x_2}, \dots, E_{x_p}$ and their inverses $E_{x_1}^{-1}, E_{x_2}^{-1}, \dots, E_{x_p}^{-1}$ subject to the relations

$$E_{x_j} \cdot g(x) = g(x + e_j) E_{x_j}, \quad (2.3)$$

for all rational functions $g(x)$ and for $j = 1, 2, \dots, p$. For every $i \in \{1, 2, \dots, p\}$ the involution I_i can be naturally *extended* to \mathcal{D}_x , by defining

$$I_i(g(x)) = g(I_i(x)), \quad I_i(E_{x_i}) = E_{x_i}^{-1}, \quad I_i(E_{x_j}) = E_{x_j} \text{ for } j \neq i. \quad (2.4)$$

It is easy to see that I_i is correctly defined because the relations (2.3) are preserved under the action of I_i , i.e. we have

$$I_i(E_{x_j}) \cdot I_i(g(x)) = I_i(g(x + e_j)) I_i(E_{x_j}),$$

for $i, j \in \{1, 2, \dots, p\}$.

For the sake of brevity we say that $L \in \mathcal{D}_x$ is I -invariant if $I_j(L) = L$ for all $j \in \{1, 2, \dots, p\}$.

2.2. The operator \mathcal{L}_p . Below we define an operator which is I -invariant. Clearly, an operator which is I -invariant is uniquely determined by its coefficient of E_x^ν (or equivalently Δ_x^ν) with $\nu_i \geq 0$ for $i = 1, 2, \dots, p$.

Let $\nu = (\nu_1, \nu_2, \dots, \nu_p) \in \{0, 1\}^p \setminus \{0\}^p$, and let $\{\nu_{i_1}, \nu_{i_2}, \dots, \nu_{i_s}\}$ be the nonzero components of ν with $0 < i_1 < i_2 < \cdots < i_s < p+1$ and $s \geq 1$. Denote

$$\begin{aligned} A_\nu &= (x_{i_1} + \beta_{i_1} - \beta_0)(x_{i_1} + \beta_{i_1}) \\ &\times \frac{\prod_{k=2}^s (x_{i_k} + x_{i_{k-1}} + \beta_{i_k})(x_{i_k} + x_{i_{k-1}} + \beta_{i_k} + 1)}{\prod_{k=1}^s (2x_{i_k} + \beta_{i_k})(2x_{i_k} + \beta_{i_k} + 1)} \\ &\times (x_{i_s} + \beta_{p+1} + N)(N - x_{i_s}). \end{aligned} \quad (2.5a)$$

An arbitrary $\nu \in \mathbb{Z}^d$ can be decomposed as $\nu = \nu^+ - \nu^-$, where $\nu^\pm \in \mathbb{N}_0^d$ with components $\nu_j^+ = \max(\nu_j, 0)$ and $\nu_j^- = -\min(\nu_j, 0)$. For $\nu \in \{-1, 0, 1\}^p \setminus \{0, 1\}^p$ we define

$$A_\nu = I^{\nu^-}(A_{\nu^+ + \nu^-}). \quad (2.5b)$$

Here I^{ν^-} is the composition of the involutions corresponding to the positive coordinates of ν^- .

Finally, we define the operator

$$\mathcal{L}_p = \mathcal{L}_p(x_1, \dots, x_p; \beta_0, \beta_1, \dots, \beta_{p+1}; N) = \sum_{\nu \in \{-1, 0, 1\}^p \setminus \{0\}^p} (-1)^{|\nu^-|} A_\nu \Delta_x^{\nu^+} \nabla_x^{\nu^-}. \quad (2.6)$$

In the above formula we use again multi-index notation, i.e. Δ_x^μ denotes the product $\prod_{j=1}^p \Delta_{x_j}^{\mu_j}$ and similarly ∇_x^μ stands for the product $\prod_{j=1}^p \nabla_{x_j}^{\mu_j}$. Since $I_i(\Delta_{x_i}) = -\nabla_{x_i}$ and $I_i(\Delta_{x_j}) = \Delta_{x_j}$ for $i \neq j$, the operator defined by equation (2.6) is I -invariant.

Lemma 2.2. *Let $L \in \mathcal{D}_x$ be an I -invariant difference operator. If*

$$\prod_{i=1}^p (2x_i + \beta_i) L(q) \in \mathcal{P} \text{ for every } q \in \mathcal{P}_\lambda,$$

then L preserves \mathcal{P}_λ , i.e. $L(\mathcal{P}_\lambda) \subset \mathcal{P}_\lambda$. In particular, the operator \mathcal{L}_p defined by (2.5)-(2.6) preserves \mathcal{P}_λ .

Proof. Let $q \in \mathcal{P}_\lambda$. Since both L and q are I -invariant, it follows that $L(q)$ is also I -invariant. Therefore, it is enough to prove that $L(q) \in \mathcal{P}$, since Remark 2.1 then implies that $L(q) \in \mathcal{P}_\lambda$. If we write $L(q)$ as a ratio of two polynomials

$$L(q) = \frac{q_1(x)}{q_2(x)},$$

where q_1 and q_2 have no common factors, then apart from a nonzero multiplicative constant, $q_2(x)$ must be the product $\prod_{i \in K} (2x_i + \beta_i)$ where $K \subset \{1, 2, \dots, p\}$. Assume that $K \neq \emptyset$ and let $i \in K$. Since $I_i(2x_i + \beta_i) = -(2x_i + \beta_i)$, we see that $I_i(q_2(x)) = -q_2(x)$ and therefore $I_i(q_1(x)) = -q_1(x)$. But it is easy to see that this condition forces $q_1(x)$ to be an odd polynomial of $(2x_i + \beta_i)$, which shows that $(2x_i + \beta_i)$ is a common factor of q_1 and q_2 leading to a contradiction.

It remains to show that \mathcal{L}_p satisfies the conditions of the Lemma. This follows easily from formulas (2.5)-(2.6) combined with the fact that $\Delta_{x_i}(\lambda_i^k)$ is divisible by $\Delta_{x_i}(\lambda_i) = (2x_i + \beta_i + 1)$, and $\nabla_{x_i}(\lambda_i^k)$ is divisible by $\nabla_{x_i}(\lambda_i) = (2x_i + \beta_i - 1)$ for every $k \in \mathbb{N}$. \square

For $d \in \mathbb{N}_0$ let us denote by \mathcal{P}_λ^d the space of polynomials in \mathcal{P}_λ of (total) degree at most d in the variables $\lambda_1, \lambda_2, \dots, \lambda_p$, with the convention that $\mathcal{P}_\lambda^{-1} = \{0\}$.

Definition 2.3. We say that a linear operator L on \mathcal{P}_λ is triangular if for every $d \in \mathbb{N}_0$ there is $c_d \in \mathbb{R}$ such that

$$L(q) = c_d q \pmod{\mathcal{P}_\lambda^{d-1}} \text{ for all } q \in \mathcal{P}_\lambda^d.$$

The main result in this section is the following proposition.

Proposition 2.4. *The operator \mathcal{L}_p defined by (2.6) is triangular. More precisely, we have*

$$\mathcal{L}_p(q) = -d(d-1+\beta_{p+1}-\beta_0)q \pmod{\mathcal{P}_\lambda^{d-1}} \text{ for every } q \in \mathcal{P}_\lambda^d. \quad (2.7)$$

Proof. For $j \in \{1, 2, \dots, p\}$ let

$$\mathcal{L}_{p,j} = \sum_{\substack{\nu \in \{-1, 0, 1\}^p \setminus \{0\}^p \\ |\nu^+| + |\nu^-| = j}} (-1)^{|\nu^-|} A_\nu \Delta_x^{\nu^+} \nabla_x^{\nu^-},$$

where A_ν are given by (2.5). Then

$$\mathcal{L}_p = \mathcal{L}_{p,1} + \mathcal{L}_{p,2} + \cdots + \mathcal{L}_{p,p}.$$

It is easy to see that for every j , the operator $\mathcal{L}_{p,j}$ is I -invariant, satisfying the conditions of Lemma 2.2, and therefore $\mathcal{L}_{p,j}$ preserves \mathcal{P}_λ . Moreover, since A_ν is a ratio of a polynomial of degree $2(|\nu^+| + |\nu^-| + 1)$ over a polynomial of degree $2(|\nu^+| + |\nu^-|)$, and since Δ_{x_i} and ∇_{x_i} decrease the total degree of a polynomial by 1, one can deduce that for $j \geq 3$ we have $\mathcal{L}_{p,j}(\mathcal{P}_\lambda^d) \subset \mathcal{P}_\lambda^{d-1}$. Thus

$$\mathcal{L}_p(q) = \mathcal{L}_{p,1}(q) + \mathcal{L}_{p,2}(q) \pmod{\mathcal{P}_\lambda^{d-1}} \text{ for every } q \in \mathcal{P}_\lambda^d, \quad (2.8)$$

i.e. we can ignore the operators $\mathcal{L}_{p,j}$ for $j \geq 3$. Clearly, it is enough to prove (2.7) for $q = \lambda^n = \lambda_1^{n_1} \lambda_2^{n_2} \cdots \lambda_p^{n_p}$. Let us write $\mathcal{L}_{p,1} = \sum_{i=1}^p \mathcal{L}_{p,1}^{(i)}$, where

$$\mathcal{L}_{p,1}^{(i)} = A_{e_i} \Delta_{x_i} - A_{-e_i} \nabla_{x_i}$$

and

$$A_{e_i} = \frac{(x_i + \beta_i - \beta_0)(x_i + \beta_i)(x_i + \beta_{p+1} + N)(N - x_i)}{(2x_i + \beta_i)(2x_i + \beta_i + 1)} \quad (2.9a)$$

$$A_{-e_i} = \frac{x_i(x_i + \beta_0)(N - x_i - \beta_i + \beta_{p+1})(N + x_i + \beta_i)}{(2x_i + \beta_i)(2x_i + \beta_i - 1)}. \quad (2.9b)$$

Using the binomial formula one can check that

$$\begin{aligned} \Delta_{x_i}(\lambda_i^{n_i}) &= (\Delta_{x_i} \lambda_i) \left(\sum_{k=0}^{n_i-1} \lambda_i^k(x_i) \lambda_i^{n_i-1-k}(x_i+1) \right) \\ &= (2x_i + \beta_i + 1)(n_i x_i^{2n_i-2} + n_i(n_i-1)(\beta_i+1)x_i^{2n_i-3} + O(x_i^{2n_i-4})) \end{aligned} \quad (2.10a)$$

$$\begin{aligned} \nabla_{x_i}(\lambda_i^{n_i}) &= (\nabla_{x_i} \lambda_i) \left(\sum_{k=0}^{n_i-1} \lambda_i^k(x_i) \lambda_i^{n_i-1-k}(x_i-1) \right) \\ &= (2x_i + \beta_i - 1)(n_i x_i^{2n_i-2} + n_i(n_i-1)(\beta_i-1)x_i^{2n_i-3} + O(x_i^{2n_i-4})). \end{aligned} \quad (2.10b)$$

From (2.9), (2.10) and the fact that the operator $\mathcal{L}_{p,1}^{(i)}$ satisfies the conditions in Lemma 2.2 one can deduce that

$$\mathcal{L}_{p,1}^{(i)}(\lambda_i^{n_i}) = (n_i(\beta_0 - \beta_{p+1} + 1) - n_i^2)x_i^{2n_i} + O(x_i^{2n_i-1}),$$

which shows that

$$\mathcal{L}_{p,1}^{(i)}(\lambda^n) = \lambda^{n-n_i e_i} \mathcal{L}_{p,1}^{(i)}(\lambda_i^{n_i}) = (n_i(\beta_0 - \beta_{p+1} + 1) - n_i^2)\lambda^n \pmod{\mathcal{P}_\lambda^{|n|-1}}.$$

Thus

$$\mathcal{L}_{p,1}(\lambda^n) = \sum_{i=1}^p \mathcal{L}_{p,1}^{(i)}(\lambda^n) = \left(|n|(\beta_0 - \beta_{p+1} + 1) - \sum_{i=1}^p n_i^2 \right) \lambda^n \pmod{\mathcal{P}_\lambda^{|n|-1}}. \quad (2.11)$$

Similarly, we can write $\mathcal{L}_{p,2} = \sum_{1 \leq i < j \leq p} \mathcal{L}_{p,2}^{(i,j)}$ where

$$\mathcal{L}_{p,2}^{(i,j)} = A_{e_i+e_j} \Delta_{x_i} \Delta_{x_j} - A_{e_i-e_j} \Delta_{x_i} \nabla_{x_j} - A_{-e_i+e_j} \nabla_{x_i} \Delta_{x_j} + A_{-e_i-e_j} \nabla_{x_i} \nabla_{x_j}.$$

Using (2.5) one can deduce as above that

$$\mathcal{L}_{p,2}^{(i,j)}(\lambda_i^{n_i} \lambda_j^{n_j}) = -2n_i n_j \lambda_i^{n_i} \lambda_j^{n_j} \pmod{\mathcal{P}_\lambda^{n_i+n_j-1}},$$

which shows that

$$\mathcal{L}_{p,2}(\lambda^n) = -2 \left(\sum_{1 \leq i < j \leq p} n_i n_j \right) \lambda^n \pmod{\mathcal{P}_\lambda^{|n|-1}}. \quad (2.12)$$

The proof follows easily from equations (2.8), (2.11) and (2.12). \square

3. MULTIVARIABLE RACAH POLYNOMIALS

In this section we show that the operator \mathcal{L}_p defined by (2.6) is self-adjoint with respect to the Racah inner product defined by Tratnik [27]. In order to unify the formulas in this section, besides the variables x_1, x_2, \dots, x_p we also use $x_0 = 0$ and $x_{p+1} = N$. Thus, in view of equation (2.1), we can put $\lambda_0 = 0$ and $\lambda_{p+1} = N(N + \beta_{p+1})$.

3.1. The operator \mathcal{L}_p in terms of the shift operators. The operator \mathcal{L}_p was defined in (2.6) in terms of the forward and the backward difference operators $\{\Delta_{x_j}, \nabla_{x_j}\}_{j=1}^p$, which was useful to prove the triangular structure of \mathcal{L}_p . Below we give the explicit form of \mathcal{L}_p in terms of the shift operators E_x^ν , which allows us to show that \mathcal{L}_p is self-adjoint with respect to the Racah inner product.

For $i \in \{0, 1, \dots, p\}$ and $(j, k) \in \{0, 1\}^2$ we define $B_i^{j,k}$ as follows

$$B_i^{0,0} = \lambda_i + \lambda_{i+1} + \frac{(\beta_i + 1)(\beta_{i+1} - 1)}{2} \quad (3.1a)$$

$$B_i^{0,1} = (x_{i+1} + x_i + \beta_{i+1})(x_{i+1} - x_i + \beta_{i+1} - \beta_i) \quad (3.1b)$$

$$B_i^{1,0} = (x_{i+1} - x_i)(x_{i+1} + x_i + \beta_{i+1}) \quad (3.1c)$$

$$B_i^{1,1} = (x_{i+1} + x_i + \beta_{i+1})(x_{i+1} + x_i + \beta_{i+1} + 1). \quad (3.1d)$$

We extend the definition of $B_i^{j,k}(z)$ for $(j, k) \in \{-1, 0, 1\}^2$ by defining

$$B_i^{-1,k} = I_i(B_i^{1,k}) \text{ for } k = 0, 1 \quad (3.1e)$$

$$B_i^{j,-1} = I_{i+1}(B_i^{j,1}) \text{ for } j = 0, 1 \quad (3.1f)$$

$$B_i^{-1,-1} = I_i(I_{i+1}(B_i^{1,1})). \quad (3.1g)$$

Next, for $i \in \{1, \dots, p\}$ we denote

$$b_i^0 = (2x_i + \beta_i + 1)(2x_i + \beta_i - 1) = 4\lambda_i + \beta_i^2 - 1 \quad (3.2a)$$

$$b_i^1 = (2x_i + \beta_i + 1)(2x_i + \beta_i) \quad (3.2b)$$

$$b_i^{-1} = I_i(b_i^1). \quad (3.2c)$$

Finally, for $\nu \in \{-1, 0, 1\}^p$ we put

$$C_\nu = 2^{p-|\nu^+|-|\nu^-|} \frac{\prod_{k=0}^p B_k^{\nu_k, \nu_{k+1}}}{\prod_{k=1}^p b_k^{\nu_k}}. \quad (3.3)$$

The main result in this subsection is the proposition below, which says that C_ν is the coefficient of E_x^ν for the operator \mathcal{L}_p .

Proposition 3.1. *The operator $\mathcal{L}_p = \mathcal{L}_p(x; \beta; N)$ can be written as*

$$\mathcal{L}_p(x; \beta; N) = \sum_{\nu \in \{-1, 0, 1\}^p} C_\nu E_x^\nu - \left(\lambda_{p+1} + \frac{(\beta_0 + 1)(\beta_{p+1} - 1)}{2} \right), \quad (3.4)$$

where C_ν are given by (3.1), (3.2) and (3.3).

Proof. From equations (3.1), (3.2) and (3.3) it follows that the operator $\mathcal{L}_p(x; \beta; N)$ defined by (3.4) is I -invariant. We prove the statement by induction on p . For $p = 1$ formulas (2.5) give

$$\begin{aligned} A_{e_1} &= \frac{(x_1 + \beta_1 - \beta_0)(x_1 + \beta_1)(x_1 + \beta_2 + N)(N - x_1)}{(2x_1 + \beta_1)(2x_1 + \beta_1 + 1)} \\ A_{-e_1} &= \frac{x_1(x_1 + \beta_0)(N - x_1 - \beta_1 + \beta_2)(N + x_1 + \beta_1)}{(2x_1 + \beta_1)(2x_1 + \beta_1 - 1)}. \end{aligned}$$

Using (3.1), (3.2) and (3.3) we get

$$\begin{aligned} C_{e_1} &= \frac{B_0^{0,1} B_1^{1,0}}{b_1^1} = \frac{(x_1 + \beta_1)(x_1 + \beta_1 - \beta_0)(N - x_1)(x_1 + \beta_2 + N)}{(2x_1 + \beta_1)(2x_1 + \beta_1 + 1)} = A_{e_1} \\ C_{-e_1} &= I_1(C_{e_1}) = A_{-e_1} \\ C_0 &= 2 \frac{B_0^{0,0} B_1^{0,0}}{b_1^0} = 2 \frac{(\lambda_1 + (\beta_0 + 1)(\beta_1 - 1)/2)(\lambda_2 + \lambda_1 + (\beta_1 + 1)(\beta_2 - 1)/2)}{4\lambda_1 + \beta_1^2 - 1}, \end{aligned}$$

where $\lambda_1 = x_1(x_1 + \beta_1)$ and $\lambda_2 = N(N + \beta_2)$. We need to check that

$$\mathcal{L}_1 = A_{e_1}(E_{x_1} - 1) - A_{-e_1}(1 - E_{x_1}^{-1}) = C_{e_1} E_{x_1} + C_{-e_1} E_{x_1}^{-1} + C_0 - \frac{\lambda_2 + (\beta_0 + 1)(\beta_2 - 1)}{2},$$

which amounts to checking that

$$-A_{e_1} - A_{-e_1} = C_0 - \frac{\lambda_2 + (\beta_0 + 1)(\beta_2 - 1)}{2}.$$

This equality can be verified by a straightforward computation using the explicit formulas for A_{e_1} , A_{-e_1} and C_0 .

Let now $p > 1$ and assume that the statement is true for $p - 1$. Let us write \mathcal{L}_p as follows

$$\begin{aligned} \mathcal{L}_p &= \mathcal{L}' \frac{(x_p + \beta_{p+1} + N)(N - x_p)}{(2x_p + \beta_p)(2x_p + \beta_p + 1)} (-\Delta_{x_p}) \\ &\quad + \mathcal{L}'' \frac{(N - x_p + \beta_{p+1} - \beta_p)(N + x_p + \beta_p)}{(2x_p + \beta_p)(2x_p + \beta_p - 1)} (\nabla_{x_p}) + \mathcal{L}''', \end{aligned} \quad (3.5)$$

where \mathcal{L}' , \mathcal{L}'' , \mathcal{L}''' are difference operators in the variables x_1, x_2, \dots, x_{p-1} with coefficients depending on x_1, \dots, x_p and the parameters $\beta_0, \beta_1, \dots, \beta_{p+1}, N$. Clearly, the operators $\mathcal{L}', \mathcal{L}'', \mathcal{L}'''$ are uniquely determined from \mathcal{L}_p . This implies that they are I_1, I_2, \dots, I_{p-1} invariant and therefore they are characterized by the coefficients of $\Delta_{\bar{x}}^{\bar{\nu}} = \Delta_{x_1}^{\nu_1} \Delta_{x_2}^{\nu_2} \cdots \Delta_{x_{p-1}}^{\nu_{p-1}}$, where $\bar{x} = (x_1, x_2, \dots, x_{p-1})$ and $\bar{\nu} = (\nu_1, \nu_2, \dots, \nu_{p-1}) \in \{0, 1\}^{p-1}$. The coefficient of $\Delta_{\bar{x}}^{\bar{\nu}}$ in \mathcal{L}' is equal to $-(x_p + \beta_p - \beta_0)(x_p + \beta_p)$ (it comes from the term $A_{e_p} \Delta_{x_p}$ in \mathcal{L}_p). Using equations (2.5)-(2.6) one can see that the coefficients of $\Delta_{\bar{x}}^{\bar{\nu}}$ for $\bar{\nu} \in \{0, 1\}^{p-1} \setminus \{\bar{0}\}$ are the same as the coefficients of

the operator $\mathcal{L}_{p-1}(x_1, \dots, x_{p-1}; \beta_0, \dots, \beta_{p-1}, \beta'_p; N')$ where $\beta'_p = 2(x_p + \beta_p) + 1$ and $N' = -x_p - \beta_p$. Thus we deduce that

$$\begin{aligned} \mathcal{L}' &= \mathcal{L}_{p-1}(x_1, \dots, x_{p-1}; \beta_0, \dots, \beta_{p-1}, 2(x_p + \beta_p) + 1; -x_p - \beta_p) \\ &\quad - (x_p + \beta_p - \beta_0)(x_p + \beta_p) \\ &= \sum_{\nu \in \{-1, 0, 1\}^{p-1}} C'_\nu E_x^\nu, \end{aligned} \tag{3.6a}$$

where C'_ν are computed from (3.1)-(3.3) for the operator \mathcal{L}_{p-1} with parameters given in (3.6a). Notice that the last term on the right-hand side of (3.4) for \mathcal{L}_{p-1} cancels with $(x_p + \beta_p - \beta_0)(x_p + \beta_p)$ in (3.6a). For \mathcal{L}'' we have

$$\mathcal{L}'' = I_p(\mathcal{L}') = \sum_{\nu \in \{-1, 0, 1\}^{p-1}} I_p(C'_\nu) E_x^\nu = \sum_{\nu \in \{-1, 0, 1\}^{p-1}} C''_\nu E_x^\nu. \tag{3.6b}$$

For the last operator \mathcal{L}''' we obtain

$$\begin{aligned} \mathcal{L}''' &= \mathcal{L}_{p-1}(x_1, \dots, x_{p-1}; \beta_0, \dots, \beta_{p-1}, \beta_{p+1}; N) \\ &= \sum_{\nu \in \{-1, 0, 1\}^{p-1}} C'''_\nu E_x^\nu - \left(\lambda_{p+1} + \frac{(\beta_0 + 1)(\beta_{p+1} - 1)}{2} \right), \end{aligned} \tag{3.6c}$$

where C'''_ν are computed from (3.1)-(3.3) for the operator \mathcal{L}_{p-1} with parameters given in (3.6c).

Notice that the last term in (3.6c) gives the last term in (3.4). Thus, it remains to show that we get the stated formulas for the coefficients C_ν in (3.4), using the decomposition (3.5). Since the operator \mathcal{L}_p is I -invariant, it is enough to prove the formulas for C_ν when ν has nonnegative coordinates. We have two possible cases depending on whether $\nu_p = 1$ or $\nu_p = 0$.

3.1.1. Case 1: $\nu_p = 1$. Write $\nu = (\nu', 1)$ with $\nu' \in \{0, 1\}^{p-1}$. From (3.5) it is clear that E_x^ν appears only in the first term on the right-hand side and we have

$$C_\nu = -C'_{\nu'} \frac{(x_p + \beta_{p+1} + N)(N - x_p)}{(2x_p + \beta_p)(2x_p + \beta_p + 1)} = -\frac{C'_{\nu'}}{b_p^1} (x_p + \beta_{p+1} + N)(N - x_p).$$

Notice that the factors $(B_i^{\nu_i, \nu_{i+1}})'$ in formula (3.3) for $C'_{\nu'}$ are the same as the factors $B_i^{\nu_i, \nu_{i+1}}$ in formula (3.3) for C_ν for $i = 0, 1, \dots, p-2$. This combined with the last formula shows that in order to complete the proof we need to check that $(B_{p-1}^{\nu_{p-1}, 0})' = -B_{p-1}^{\nu_{p-1}, 1}$.

Again we have two possibilities:

If $\nu_{p-1} = 0$ then from the parameters in formula (3.6a) we get

$$\begin{aligned} (B_{p-1}^{0,0})' &= \lambda_{p-1} + N'(N' + \beta'_p) + \frac{(\beta_{p-1} + 1)(\beta'_p - 1)}{2} \\ &= x_{p-1}(x_{p-1} + \beta_{p-1}) - (x_p + \beta_p)(x_p + \beta_p + 1) + (\beta_{p-1} + 1)(x_p + \beta_p) \end{aligned}$$

On the other hand from (3.1b) we have

$$B_{p-1}^{0,1} = (x_p + x_{p-1} + \beta_p)(x_p - x_{p-1} + \beta_p - \beta_{p-1}).$$

From the last two formulas it is easy to see that $(B_{p-1}^{0,0})' = -B_{p-1}^{0,1}$.

If $\nu_{p-1} = 1$ then

$$(B_{p-1}^{1,0})' = (N' - x_{p-1})(N' + x_{p-1} + \beta'_p) = -(x_p + x_{p-1} + \beta_p)(x_p + \beta_p + 1 + x_{p-1}) = -B_{p-1}^{11},$$

completing the proof in the case $\nu_p = 1$.

3.1.2. *Case 2:* $\nu_p = 0$. Let us write again $\nu = (\nu', 0)$ with $\nu' \in \{0, 1\}^{p-1}$. Then from (3.5) we deduce

$$\begin{aligned} C_\nu &= C'_{\nu'} \frac{(x_p + \beta_{p+1} + N)(N - x_p)}{(2x_p + \beta_p)(2x_p + \beta_p + 1)} \\ &\quad + C''_{\nu'} \frac{(N - x_p + \beta_{p+1} - \beta_p)(N + x_p + \beta_p)}{(2x_p + \beta_p)(2x_p + \beta_p - 1)} + C'''_{\nu'}. \end{aligned}$$

We need to check formula (3.3) for C_ν . Again the factors $B_i^{\nu_i, \nu_{i+1}}$ for $i = 0, 1, \dots, p-2$ and the denominator factors $b_i^{\nu_i}$ for $i = 1, 2, \dots, p-1$ are common for C_ν , $C'_{\nu'}$, $C''_{\nu'}$ and $C'''_{\nu'}$. Thus we need to verify that

$$\begin{aligned} \frac{2B_{p-1}^{\nu_{p-1}, 0} B_p^{0, 0}}{(2x_p + \beta_p + 1)(2x_p + \beta_p - 1)} &= (B_{p-1}^{\nu_{p-1}, 0})' \frac{(x_p + \beta_{p+1} + N)(N - x_p)}{(2x_p + \beta_p)(2x_p + \beta_p + 1)} \\ &\quad + (B_{p-1}^{\nu_{p-1}, 0})'' \frac{(N - x_p + \beta_{p+1} - \beta_p)(N + x_p + \beta_p)}{(2x_p + \beta_p)(2x_p + \beta_p - 1)} + (B_{p-1}^{\nu_{p-1}, 0})'''. \end{aligned}$$

Using the explicit formulas (3.1) and considering separately the two possible cases $\nu_{p-1} = 0$ and $\nu_{p-1} = 1$ one can check that the above equality holds, thus completing the proof. \square

3.2. Racah inner product and polynomials. For a positive integer N we define an inner product on the space \mathcal{P}_λ^N by

$$\langle f, g \rangle = \sum_{x \in V_N} f(x)g(x)\rho(x), \quad (3.7)$$

where $f, g \in \mathcal{P}_\lambda^N$, V_N is the set

$$V_N = \{x \in \mathbb{N}_0^p : 0 = x_0 \leq x_1 \leq x_2 \leq \dots \leq x_p \leq x_{p+1} = N\}$$

and the weight $\rho(x)$ is given by

$$\rho(x) = \prod_{k=0}^p \frac{\Gamma(\beta_{k+1} - \beta_k + x_{k+1} - x_k)\Gamma(\beta_{k+1} + x_{k+1} + x_k)}{(x_{k+1} - x_k)!\Gamma(\beta_k + 1 + x_{k+1} + x_k)} \prod_{k=1}^p (\beta_k + 2x_k). \quad (3.8)$$

Remark 3.2. If we put $\beta_0 = \alpha_1 - \eta - 1$, $\beta_k = \sum_{j=1}^k \alpha_j$ for $k = 1, 2, \dots, p+1$, $\gamma = -N-1$, then one can check that the weight defined in (3.8) differs by a constant factor (independent of x) from the weight used by Tratnik, see [27, formula (2.3) on page 2338].

In [27] one finds an explicit orthogonal basis in \mathcal{P}_λ^N . If we denote by $r_n(\alpha, \beta, \gamma, \delta; x)$ the one dimensional Racah polynomials

$$\begin{aligned} r_n(\alpha, \beta, \gamma, \delta; x) &= (\alpha + 1)_n (\beta + \delta + 1)_n (\gamma + 1)_n \\ &\quad \times {}_4F_3 \left[\begin{matrix} -n, n + \alpha + \beta + 1, -x, x + \gamma + \delta + 1 \\ \alpha + 1, \beta + \delta + 1, \gamma + 1 \end{matrix}; 1 \right], \end{aligned} \quad (3.9)$$

then an orthogonal basis in \mathcal{P}_λ^N is given by the polynomials

$$R_p(n; x; \beta; N) = \prod_{k=1}^p r_{n_k} (2N_1^{k-1} + \beta_k - \beta_0 - 1, \beta_{k+1} - \beta_k - 1, N_1^{k-1} - x_{k+1} - 1, \\ N_1^{k-1} + \beta_k + x_{k+1}; -N_1^{k-1} + x_k). \quad (3.10)$$

Here $\beta = (\beta_0, \beta_1, \dots, \beta_{p+1})$ are the parameters appearing in the weight (3.8), $n = (n_1, n_2, \dots, n_p) \in \mathbb{N}_0^p$ is such that $|n| \leq N$ and $N_1^j = n_1 + n_2 + \dots + n_j$ with $N_1^0 = 0$.

Remark 3.3. Notice that the k -th term in the product on the right-hand side in (3.10) is

$$r_{n_k} = (2N_1^{k-1} + \beta_k - \beta_0)_{n_k} (N_1^{k-1} - x_{k+1})_{n_k} (N_1^{k-1} + \beta_{k+1} + x_{k+1})_{n_k} \\ \times {}_4F_3 \left[\begin{matrix} -n_k, n_k + 2N_1^{k-1} + \beta_{k+1} - \beta_0 - 1, N_1^{k-1} - x_k, N_1^{k-1} + \beta_k + x_k \\ 2N_1^{k-1} + \beta_k - \beta_0, N_1^{k-1} + \beta_{k+1} + x_{k+1}, N_1^{k-1} - x_{k+1} \end{matrix}; 1 \right]. \quad (3.11)$$

From this formula it is easy to see that each r_{n_k} is an I -invariant polynomial of x and therefore $R_p(n; x; \beta; N) \in \mathcal{P}_\lambda$.

3.3. Self-adjointness of \mathcal{L}_p . For a difference operator $L = \sum_{\nu \in S} C_\nu(x) E^\nu$ (where S is a finite subset of \mathbb{Z}^p) we can always assume that S is symmetric about the origin, by adding finitely many coefficients $C_\nu(x)$ identically equal to zero. In the next lemma we show that simple conditions relating $C_\nu(x)$ and $C_{-\nu}(x)$ for every $\nu \neq 0$ and certain boundary conditions allow us to check whether the operator L is self-adjoint with respect to the inner product (3.7).

Lemma 3.4. *If an operator $L = \sum_{\nu \in S} C_\nu(x) E^\nu$ satisfies the following conditions for every $0 \neq \nu \in S$*

- (i) $\rho(x)C_\nu(x) = \rho(x + \nu)C_{-\nu}(x + \nu)$ when $x, x + \nu \in V_N$;
- (ii) $C_\nu(x) = 0$ when $x \in V_N$ but $x + \nu \notin V_N$

then L is self-adjoint with respect to the inner product (3.7).

Proof. For $0 \neq \nu \in S$ let $L_\nu = C_\nu(x)E^\nu + C_{-\nu}(x)E^{-\nu}$. Then

$$\langle L_\nu f, g \rangle = \sum_{x \in V_N} C_\nu(x) f(x + \nu) g(x) \rho(x) + \sum_{x \in V_N} C_{-\nu}(x) f(x - \nu) g(x) \rho(x)$$

which becomes, using condition (ii),

$$= \sum_{x: x \in V_N, x + \nu \in V_N} C_\nu(x) f(x + \nu) g(x) \rho(x) \\ + \sum_{x: x \in V_N, x - \nu \in V_N} C_{-\nu}(x) f(x - \nu) g(x) \rho(x),$$

replacing x by $x - \nu$ in the first sum and x by $x + \nu$ in the second

$$= \sum_{x: x - \nu \in V_N, x \in V_N} C_\nu(x - \nu) f(x) g(x - \nu) \rho(x - \nu) \\ + \sum_{x: x + \nu \in V_N, x \in V_N} C_{-\nu}(x + \nu) f(x) g(x + \nu) \rho(x + \nu),$$

then using condition (i)

$$\begin{aligned} &= \sum_{x: x-\nu \in V_N, x \in V_N} C_{-\nu}(x) f(x) g(x-\nu) \rho(x) \\ &\quad + \sum_{x: x \in V_N, x+\nu \in V_N} C_\nu(x) f(x) g(x+\nu) \rho(x), \end{aligned}$$

again using condition (ii)

$$\begin{aligned} &= \sum_{x \in V_N} C_{-\nu}(x) f(x) g(x-\nu) \rho(x) + \sum_{x \in V_N} C_\nu(x) f(x) g(x+\nu) \rho(x) \\ &= \langle f, L_\nu g \rangle. \end{aligned}$$

The proof follows immediately by writing L as a sum of L_ν 's. \square

Lemma 3.5. *For $N \in \mathbb{N}$, the operator $\mathcal{L}_p(x; \beta; N)$ is self-adjoint with respect to the inner product defined by (3.7)-(3.8).*

Proof. Using Proposition 3.1 we show that the conditions in Lemma 3.4 are satisfied. Let us write $\rho(x)$ in (3.8) as

$$\rho(x) = \prod_{k=0}^p \rho'_k(x) \prod_{k=1}^p \rho''_k(x),$$

where

$$\rho'_k(x) = \frac{\Gamma(\beta_{k+1} - \beta_k + x_{k+1} - x_k) \Gamma(\beta_{k+1} + x_{k+1} + x_k)}{(x_{k+1} - x_k)! \Gamma(\beta_k + 1 + x_{k+1} + x_k)} \quad (3.12a)$$

$$\rho''_k(x) = \beta_k + 2x_k. \quad (3.12b)$$

Then condition (i) in Lemma 3.4 will follow if we show that for $x, x+\nu \in V_N$ we have

$$\frac{\rho'_k(x+\nu)}{\rho'_k(x)} = \frac{B_k^{\nu_k, \nu_{k+1}}(x)}{B_k^{-\nu_k, -\nu_{k+1}}(x+\nu)} \quad (3.13a)$$

and

$$\frac{\rho''_k(x+\nu)}{\rho''_k(x)} = \frac{b_k^{-\nu_k}(x+\nu)}{b_k^{\nu_k}(x)}. \quad (3.13b)$$

Notice that equations (3.13) for ν and $-\nu$ are essentially the same. Thus it is enough to check (3.13a) for $(\nu_k, \nu_{k+1}) = (0, 0), (1, 0), (0, 1), (1, 1), (1, -1)$, while (3.13b) needs to be verified for $\nu_k = 0, 1$.

Let us start with (3.13b). If $\nu_k = 0$ then clearly both sides of (3.13b) are equal to 1. If $\nu_k = 1$ then from (3.2) we see that $b_k^1 = (2x_k + \beta_k)(2x_k + \beta_k + 1)$ and $b_k^{-1} = (2x_k + \beta_k)(2x_k + \beta_k - 1)$ which combined with (3.12b) shows that both sides of (3.13b) are equal to $(2x_k + \beta_k + 2)/(2x_k + \beta_k)$.

Next we verify equation (3.13a).

Case 1: $\nu_k = \nu_{k+1} = 0$. Clearly both sides of (3.13a) are equal to 1.

Case 2: $\nu_k = 1, \nu_{k+1} = 0$. From (3.1) we see that

$$B_k^{1,0} = (x_{k+1} - x_k)(x_{k+1} + x_k + \beta_{k+1}) \quad (3.14a)$$

$$B_k^{-1,0} = I_k(B_k^{1,0}) = (x_{k+1} + x_k + \beta_k)(x_{k+1} - x_k + \beta_{k+1} - \beta_k). \quad (3.14b)$$

Using the well known property of the gamma function

$$\Gamma(x) = (x - n)_n \Gamma(x - n), \quad (3.15)$$

one can easily verify that the left-hand side of (3.13a) gives the same ratio as $B_k^{1,0}(x)/B_k^{-1,0}(x + \nu)$.

Case 3: $\nu_k = 0, \nu_{k+1} = 1$. In this case we have

$$B_k^{0,1} = (x_{k+1} + x_k + \beta_{k+1})(x_{k+1} - x_k + \beta_{k+1} - \beta_k) \quad (3.16a)$$

$$B_k^{-1,-1} = I_{k+1}(B_k^{1,0}) = (x_{k+1} - x_k)(x_{k+1} + x_k + \beta_k), \quad (3.16b)$$

and applying (3.15) to the left-hand side of (3.13a) one can check that it is equal to $B_k^{0,1}(x)/B_k^{-1,-1}(x + \nu)$.

Case 4: $\nu_k = \nu_{k+1} = 1$. This time we need

$$B_k^{1,1} = (x_{k+1} + x_k + \beta_{k+1})(x_{k+1} + x_k + \beta_{k+1} + 1) \quad (3.17a)$$

$$B_k^{-1,-1} = I_{k+1}(I_k(B_k^{1,1})) = (x_{k+1} + x_k + \beta_k)(x_{k+1} + x_k + \beta_k - 1), \quad (3.17b)$$

and the verification of (3.13a) goes along the same lines.

Case 5: $\nu_k = 1, \nu_{k+1} = -1$. In this last case we use

$$B_k^{1,-1} = I_{k+1}(B_k^{1,1}) = (x_{k+1} - x_k)(x_{k+1} - x_k - 1) \quad (3.18a)$$

$$B_k^{-1,1} = I_k(B_k^{1,1}) = (x_{k+1} - x_k + \beta_{k+1} - \beta_k)(x_{k+1} - x_k + \beta_{k+1} - \beta_k + 1), \quad (3.18b)$$

and the verification of (3.13a) can be done as in the previous cases.

Finally, we need to check condition (ii) in Lemma 3.4. Let $0 \neq \nu \in \{-1, 0, 1\}^p$ and let $x \in V_N$ such that $x + \nu \notin V_N$. Since $x \in V_N$ it follows that $x_i \leq x_{i+1}$ for every $i \in \{0, 1, \dots, p\}$. The condition $x + \nu \notin V_N$ means that for some $k \in \{0, 1, \dots, p\}$ we have $x_k + \nu_k > x_{k+1} + \nu_{k+1}$. Here again we use the convention that $\nu_0 = \nu_{p+1} = 0$. Since $x_k \leq x_{k+1}$ the last inequality implies that $\nu_k > \nu_{k+1}$. Thus (ν_k, ν_{k+1}) must be one of the pairs: $(0, -1)$, $(1, 0)$, $(1, -1)$.

If $\nu_k = 0, \nu_{k+1} = -1$ then we must have that $x_k = x_{k+1}$. Since $C_\nu(x)$ contains the factor $B_k^{0,-1}$ which is 0 when $x_k = x_{k+1}$ (see (3.16b)), we conclude that $C_\nu(x) = 0$.

If $\nu_k = 1, \nu_{k+1} = 0$ then again we must have that $x_k = x_{k+1}$. This time $C_\nu(x)$ contains the factor $B_k^{1,0}$ which is 0 when $x_k = x_{k+1}$ (see (3.14a)) and therefore $C_\nu(x) = 0$.

Finally, if $\nu_k = 1, \nu_{k+1} = -1$ then $x_k = x_{k+1}$ or $1 + x_k = x_{k+1}$. In both cases, $B_k^{1,-1} = 0$ (see (3.18a)) and therefore $C_\nu(x) = 0$. \square

3.4. Admissibility of \mathcal{L}_p and the commutative algebra \mathcal{A}_x . As an immediate corollary of Proposition 2.4 and Lemma 3.5 we obtain the following theorem.

Theorem 3.6. *Let $N \in \mathbb{N}_0$ and let $\{Q(n; \lambda(x); \beta; N) : |n| \leq N\}$ be a family of polynomials that span \mathcal{P}_λ^N such that $Q(n; \lambda(x); \beta; N)$ is a polynomial of total degree $|n|$ which is orthogonal to polynomials of degree at most $|n| - 1$ with respect to the inner product (3.7)-(3.8). Then $Q(n; \lambda(x); \beta; N)$ are eigenfunctions of the operator $\mathcal{L}_p(x; \beta; N)$ with eigenvalue $-|n|(|n| - 1 + \beta_{p+1} - \beta_0)$.*

Remark 3.7. Following [30] we can consider the ideal

$$\mathfrak{I}(V_N) = \{q \in \mathcal{P}_\lambda : q(x) = 0 \text{ for every } x \in V_N\}.$$

If we denote

$$\sigma_n(x) = (-x_1)_{n_1}(-x_2 + n_1)_{n_2} \cdots (-x_p + n_1 + n_2 + \cdots + n_{p-1})_{n_p},$$

then it is easy to see that $\sigma_n(x) = 0$ when $|n| \geq N + 1$ and $x \in V_N$. This shows that the I -invariant set

$$\begin{aligned} \mathfrak{S}_N = & \{(-x_1)_{n_1}(x_1 + \beta_1)_{n_1}(-x_2 + n_1)_{n_2}(x_2 + \beta_2 + n_1)_{n_2} \cdots \\ & \times (-x_p + n_1 + \cdots + n_{p-1})_{n_p}(x_p + \beta_p + n_1 + \cdots + n_{p-1})_{n_p} : |n| = N + 1\} \end{aligned}$$

is contained in $\mathcal{I}(V_N)$ and therefore for every $q \in \mathcal{P}_\lambda$ there exists $\bar{q} \in \mathcal{P}_\lambda^N$ such that $q - \bar{q} \in \mathcal{I}(V_N)$. If $n \in \mathbb{N}_0^p$ is such that $|n| \leq N$ then the polynomial $R_p(n; x; \beta; N)$ defined by (3.10) has a positive norm which means that it cannot be identically equal to zero on V_N . Moreover, since $\{R_p(n; x; \beta; N) : |n| \leq N\}$ are mutually orthogonal, we see that there is no polynomial (other than 0) of degree at most N belonging to $\mathcal{I}(V_N)$. Thus the ideal $\mathcal{I}(V_N)$ is generated by the set \mathfrak{S}_N which allows us to identify the factor space $\mathcal{P}_\lambda/\mathcal{I}(V_N)$ with \mathcal{P}_λ^N .

Theorem 3.6 shows that the operator \mathcal{L}_p is admissible on V_N because for every $k \in \{0, 1, \dots, N\}$ the equation $\mathcal{L}_p u = -k(k-1+\beta_{p+1}-\beta_0)u$ has $\binom{k+p-1}{p-1} = \dim(\mathcal{P}_\lambda^k/\mathcal{P}_\lambda^{k-1})$ linearly independent solutions in \mathcal{P}_λ^k and it has no nontrivial solutions in the space $\mathcal{P}_\lambda^{k-1}$, see [16, Definition 3.3].

Next we focus on the polynomials $R_p(n; x; \beta; N)$ defined by (3.10) for arbitrary $(\beta, N) \in \mathbb{R}^{p+3}$, and we construct a commutative algebra generated by p difference operators which are diagonalized by $R_p(n; x; \beta; N)$. Before we state the main theorem, we formulate a technical lemma, where we prove the intuitively clear statement that a difference operator vanishing on \mathcal{P}_λ must be the zero operator.

Lemma 3.8. *If $L = \sum_{\nu \in S} L_\nu(x) E_x^\nu$ is a difference operator in \mathbb{R}^p , with rational coefficients, such that $L(q) = 0$ for every $q \in \mathcal{P}_\lambda$, then L is the zero operator, i.e. $L_\nu(x) = 0$ for every $\nu \in S$.*

Proof. Without any restriction we can assume that $S \subset \mathbb{N}_0^p$. Indeed if we fix $l \in \mathbb{N}_0^p$, then L is the zero operator if and only if $E_x^l \cdot L$ is the zero operator. Thus, it is enough to prove the statement for the operator $E_x^l \cdot L$ and choosing the coordinates of l large enough, this operator has only nonnegative powers of E_{x_j} .

Next, we can replace E_{x_j} by $\Delta_{x_j} + 1$, which shows that every operator $L = \sum_{\nu \in S} L_\nu(x) E_x^\nu$ with $S \subset \mathbb{N}_0^p$ can be uniquely written as

$$L = \sum_{\nu \in S'} L'_\nu(x) \frac{\Delta_x^\nu}{\nu!} = \sum_{\substack{\nu \in \mathbb{N}_0^p \\ |\nu| \leq M}} L'_\nu(x) \frac{\Delta_x^\nu}{\nu!},$$

where $M \in \mathbb{N}$ is large enough so that $S' \subset S_M = \{\nu \in \mathbb{N}_0^p : |\nu| \leq M\}$ and we have set $L'_\nu(x) = 0$ for $\nu \notin S'$.

Now we use the fact that $L(\lambda^m) = 0$ for all $m \in S_M$. We obtain a system of linear equations for $L'_\nu(x)$ with determinant

$$D = \det_{\nu, m \in S_M} \left[\frac{\Delta_x^\nu}{\nu!} \lambda^m \right].$$

We prove below that

$$D = (2^p x_1 x_2 \cdots x_p)^{\binom{M+p}{p+1}} + \text{polynomial in } x \text{ of total degree} < p \binom{M+p}{p+1}, \quad (3.19)$$

which shows that D is a nonzero polynomial and therefore all coefficients $L_\nu(x)$ must be equal to 0.

Notice that

$$\frac{\Delta_x^\nu}{\nu!} x^{2m} = \binom{2m}{\nu} x^{2m-\nu} + \text{polynomial in } x \text{ of total degree } < 2|m| - |\nu|.$$

Moreover,

$$\det_{\nu, m \in S_M} \left[\binom{2m}{\nu} x^{2m-\nu} \right] = \det_{\nu, m \in S_M} \left[\binom{2m}{\nu} \right] (x_1 x_2 \cdots x_p)^d,$$

where

$$d = \sum_{j=1}^M j \binom{M-j+p-1}{p-1} = \binom{M+p}{p+1}.$$

Thus, to prove (3.19) it remains to show that

$$\det_{\nu, m \in S_M} \left[\binom{2m}{\nu} \right] = 2^{p(\frac{M+p}{p+1})}. \quad (3.20)$$

Let us denote $\mathcal{C}(M, p+1) = \{\alpha \in \mathbb{N}_0^{p+1} : |\alpha| = M\}$. Then by [6, Theorem 5, p. 357] for every $z, l \in \mathbb{R}^{p+1}$ we have

$$\begin{aligned} \det_{\alpha, \alpha' \in \mathcal{C}(M, p+1)} \left[\binom{z+l\alpha}{\alpha'} \right] &= \left(\prod_{j=1}^{p+1} l_j \right)^{\binom{M+p}{p+1}} \\ &\times \frac{\prod_{\epsilon_i \geq 0, \epsilon_1 + \epsilon_2 + \cdots + \epsilon_{p+1} < M} \left(M + \sum_{j=1}^{p+1} \left(\frac{z_j}{l_j} - \frac{\epsilon_j}{l_j} \right) \right)}{\prod_{i=1}^M i^{\binom{M+p-i}{p}}}, \end{aligned}$$

where $l\alpha$ denotes $(l_1\alpha_1, l_2\alpha_2, \dots, l_{p+1}\alpha_{p+1})$. If we apply this identity with $\alpha = (m, M-|m|)$, $\alpha' = (\nu, M-|\nu|)$, $z_1 = z_2 = \cdots = z_p = 0$, $l_1 = l_2 = \cdots = l_p = 2$, $l_{p+1} = 0$ we obtain

$$\det_{\nu, m \in S_M} \left[\binom{2m}{\nu} \binom{z_{p+1}}{M-|\nu|} \right] = 2^{p(\frac{M+p}{p+1})} \frac{\prod_{\epsilon_i \geq 0, \epsilon_1 + \epsilon_2 + \cdots + \epsilon_{p+1} < M} (z_{p+1} - \epsilon_{p+1})}{\prod_{i=1}^M i^{\binom{M+p-i}{p}}}. \quad (3.21)$$

One can check now that

$$\det_{\nu, m \in S_M} \left[\binom{2m}{\nu} \binom{z_{p+1}}{M-|\nu|} \right] = \prod_{k=1}^M \binom{z_{p+1}}{k}^{\binom{M-k+p-1}{p-1}} \det_{\nu, m \in S_M} \left[\binom{2m}{\nu} \right] \quad (3.22)$$

and

$$\begin{aligned} \prod_{k=1}^M \binom{z_{p+1}}{k}^{\binom{M-k+p-1}{p-1}} &= \frac{\prod_{j=0}^{M-1} (z_{p+1} - j)^{\binom{M-j+p-1}{p}}}{\prod_{i=1}^M i^{\binom{M+p-i}{p}}} \\ &= \frac{\prod_{\epsilon_i \geq 0, \epsilon_1 + \epsilon_2 + \cdots + \epsilon_{p+1} < M} (z_{p+1} - \epsilon_{p+1})}{\prod_{i=1}^M i^{\binom{M+p-i}{p}}}. \end{aligned} \quad (3.23)$$

Equation (3.20) follows immediately from (3.21), (3.22) and (3.23), thus completing the proof. \square

From now on, we use the following convention:

- when $N \notin \mathbb{N}_0$ we consider the polynomials $R_p(n; x; \beta; N)$ for all $n \in \mathbb{N}_0^p$;
- when $N \in \mathbb{N}_0$ we consider the polynomials $R_p(n; x; \beta; N)$ for $n \in \mathbb{N}_0^p$ such that $|n| \leq N$.

Theorem 3.9. For $(\beta, N) \in \mathbb{R}^{p+3}$ and for $j \in \{1, 2, \dots, p\}$ we define

$$\mathfrak{L}_j^x = \mathcal{L}_j(x_1, \dots, x_j; \beta_0, \dots, \beta_{j+1}; x_{j+1}) \quad (3.24a)$$

$$\mu_j(n) = -(n_1 + n_2 + \dots + n_j)(n_1 + n_2 + \dots + n_j - 1 + \beta_{j+1} - \beta_0). \quad (3.24b)$$

Then

$$\mathfrak{L}_j^x R_p(n; x; \beta; N) = \mu_j(n) R_p(n; x; \beta; N) \quad (3.25)$$

for every $j \in \{1, 2, \dots, p\}$ and the operators \mathfrak{L}_j^x commute with each other, i.e. $\mathcal{A}_x = \mathbb{R}[\mathfrak{L}_1^x, \mathfrak{L}_2^x, \dots, \mathfrak{L}_p^x]$ is a commutative subalgebra of \mathcal{D}_x .

Proof. If $N \in \mathbb{N}_0$ equation (3.25) follows from Theorem 3.6. Indeed, the product of the first j terms on the right-hand side of (3.10) is precisely

$$R_j(n_1, \dots, n_j; x_1, \dots, x_j; \beta_0, \dots, \beta_{j+1}; x_{j+1}),$$

while the remaining terms do not depend on the variables x_1, \dots, x_j .

Notice next that $R_p(n; x; \beta; N)$ is a polynomial in N . Thus, if we fix $n \in \mathbb{N}_0^p$, then both sides of (3.25) are polynomials in N . Since (3.25) holds for every $N \in \mathbb{N}_0$ such that $N \geq |n|$ we conclude that it must be true for every $N \in \mathbb{R}$.

Finally, from (3.25) it follows that for $i, j \in \{1, 2, \dots, p\}$

$$[\mathfrak{L}_i^x, \mathfrak{L}_j^x] R_p(n; x; \beta; N) = 0, \text{ for all } n \in \mathbb{N}_0^p.$$

Thus, if $N \notin \mathbb{N}_0$ then $[\mathfrak{L}_i^x, \mathfrak{L}_j^x]q = 0$ for every $q \in \mathcal{P}_\lambda$ which combined with Lemma 3.8 shows that $[\mathfrak{L}_i^x, \mathfrak{L}_j^x] = 0$. Since the coefficients of the operator $[\mathfrak{L}_i^x, \mathfrak{L}_j^x]$ are polynomials of N , it follows that if \mathfrak{L}_i^x and \mathfrak{L}_j^x commute for $N \notin \mathbb{N}_0$ they will also commute for $N \in \mathbb{N}_0$ which completes the proof. \square

Remark 3.10. From (2.5) it follows that the operator \mathcal{L}_p defined in (2.6) is invariant under the change $N \rightarrow -N - \beta_{p+1}$. This shows that all operators \mathfrak{L}_j^x for $j = 1, 2, \dots, p$, given in (3.24a) are I -invariant, i.e. the whole algebra \mathcal{A}_x is I -invariant.

4. BISPECTRALITY

4.1. Duality. Let $x = (x_1, x_2, \dots, x_p) \in \mathbb{R}^p$, $n = (n_1, n_2, \dots, n_p) \in \mathbb{R}^p$, $\beta = (\beta_0, \beta_1, \dots, \beta_{p+1}) \in \mathbb{R}^{p+2}$ and let us define dual variables $\tilde{x} \in \mathbb{R}^p$, $\tilde{n} \in \mathbb{R}^p$ and dual parameters $\tilde{\beta} \in \mathbb{R}^{p+2}$ by

$$x_i = \tilde{N}_1^{p+1-i} + \tilde{\beta}_{p+2-i} - \tilde{\beta}_0 + N - 1 \text{ for } i = 1, \dots, p, \quad (4.1a)$$

$$n_1 = \tilde{x}_p + \tilde{\beta}_p + N, \quad (4.1b)$$

$$n_i = \tilde{x}_{p+1-i} - \tilde{x}_{p+2-i} + \tilde{\beta}_{p+1-i} - \tilde{\beta}_{p+2-i} \text{ for } i = 2, \dots, p, \quad (4.1c)$$

$$\beta_0 = \tilde{\beta}_0 \quad (4.1d)$$

$$\beta_i = \tilde{\beta}_0 - \tilde{\beta}_{p+2-i} - 2N + 1, \text{ for } i = 1, \dots, p+1 \quad (4.1e)$$

with the usual convention that $\tilde{N}_1^j = \tilde{n}_1 + \tilde{n}_2 + \dots + \tilde{n}_j$.

Lemma 4.1. The mapping $\mathfrak{f} : (\tilde{x}, \tilde{n}, \tilde{\beta}) \rightarrow (x, n, \beta)$ given by (4.1) is a bijection $\mathbb{R}^{3p+2} \rightarrow \mathbb{R}^{3p+2}$ such that $\mathfrak{f}^{-1} = \mathfrak{f}$.

Proof. From equations (4.1d)-(4.1e) one can easily deduce that mapping

$$(\tilde{\beta}_0, \tilde{\beta}_1, \dots, \tilde{\beta}_{p+1}) \rightarrow (\beta_0, \beta_1, \dots, \beta_{p+1})$$

is one-to-one, and that the inverse is given by the same formulas. From (4.1b) and (4.1c) we find

$$N_1^i = \tilde{x}_{p+1-i} + \tilde{\beta}_{p+1-i} + N, \quad \text{for } i = 1, 2, \dots, p, \quad (4.2)$$

which shows that \tilde{x} can be written in terms of n and β by a formula analogous to (4.1a). From (4.1a) with $i = p$ we obtain $\tilde{n}_1 = x_p - N - \tilde{\beta}_2 - \tilde{\beta}_0 + 1 = x_p + N + \beta_p$. Finally, subtracting (4.1a) for i and $i+1$ we get $x_i - x_{i+1} = \tilde{n}_{p+1-i} + \beta_{p+2-i} - \tilde{\beta}_{p+1-i}$ which leads to a formula analogous to (4.1c) for \tilde{n}_i . \square

Lemma 4.2. *For $k = 1, p$ the polynomial r_{n_k} in equation (3.11) can be rewritten as*

$$\begin{aligned} r_{n_k} &= (1 - N_1^k - x_k - \beta_k)_{n_k} (\beta_{k+1} - \beta_k)_{n_k} (-N_1^k - \beta_k + \beta_0 + 1 - x_k)_{n_k} \times \\ &\quad {}_4F_3 \left[\begin{matrix} -n_k, -x_{k+1} - x_k - \beta_k, x_{k+1} - x_k + \beta_{k+1} - \beta_k, 1 - 2N_1^{k-1} - n_k - \beta_k + \beta_0 \\ 1 - N_1^k - x_k - \beta_k, \beta_{k+1} - \beta_k, -N_1^k - \beta_k + \beta_0 + 1 - x_k \end{matrix} ; 1 \right] \end{aligned} \quad (4.3a)$$

and for $k = 2, 3, \dots, p-1$ we can write r_{n_k} as

$$\begin{aligned} r_{n_k} &= (1 - N_1^k - x_k - \beta_k)_{n_k} (N_1^{k-1} + x_{k+1} + \beta_{k+1} - \beta_0)_{n_k} (x_{k+1} - x_k - n_k + 1)_{n_k} \times \\ &\quad \times {}_4F_3 \left[\begin{matrix} -n_k, x_{k+1} - x_k + \beta_{k+1} - \beta_k, N_1^{k-1} - \beta_0 - x_k, 1 - N_1^k + x_{k+1} \\ 1 - N_1^k - x_k - \beta_k, N_1^{k-1} + x_{k+1} + \beta_{k+1} - \beta_0, x_{k+1} - x_k - n_k + 1 \end{matrix} ; 1 \right]. \end{aligned} \quad (4.3b)$$

Proof. First we iterate the identity of Whipple (see [3, p. 56]) connecting the terminating Saalschützian ${}_4F_3$

$$\begin{aligned} (u)_n (v)_n (w)_n {}_4F_3 \left[\begin{matrix} -n, x, y, z \\ u, v, w \end{matrix} ; 1 \right] \\ = (1 - v - z - n)_n (1 - w + z - n)_n (u)_n {}_4F_3 \left[\begin{matrix} -n, u - x, u - y, z \\ 1 - v + z - n, 1 - w + z - n, u \end{matrix} ; 1 \right]. \end{aligned} \quad (4.4)$$

Using this transformation again on the right-hand side of (4.4) with z and u replaced by $u - x$ and $1 - v + z - n$, respectively, yields

$$\begin{aligned} (u)_n (v)_n (w)_n {}_4F_3 \left[\begin{matrix} -n, x, y, z \\ u, v, w \end{matrix} ; 1 \right] \\ = (1 - x - n)_n (1 - v + y - n)_n (1 - v + z - n)_n \times {}_4F_3 \left[\begin{matrix} -n, w - x, u - x, 1 - v - n \\ 1 - x - n, 1 - v + y - n, 1 - v + z - n \end{matrix} ; 1 \right]. \end{aligned} \quad (4.5)$$

Applying (4.5) with $x = N_1^{k-1} + \beta_k + x_k$, $y = n_k + 2N_1^{k-1} + \beta_{k+1} - \beta_0 - 1$, $z = N_1^{k-1} - x_k$, $v = 2N_1^{k-1} + \beta_k - \beta_0$, $u = N_1^{k-1} + \beta_{k+1} + x_{k+1}$, $w = N_1^{k-1} - x_{k+1}$ we find for $k = 1, p$ equation (4.3a).

When $k = 2, 3, \dots, p-1$ we choose $x = N_1^{k-1} + \beta_k + x_k$, $y = n_k + 2N_1^{k-1} + \beta_{k+1} - \beta_0 - 1$, $z = N_1^{k-1} - x_k$, $v = N_1^{k-1} - x_{k+1}$, $u = 2N_1^{k-1} + \beta_k - \beta_0$, $w = N_1^{k-1} + \beta_{k+1} + x_{k+1}$ and we obtain equation (4.3b). \square

Using (4.1)-(4.2) one can deduce the following lemma.

Lemma 4.3. *If (x, n, β) and $(\tilde{x}, \tilde{n}, \tilde{\beta})$ are related by (4.1) and if $n \in \mathbb{N}_0^p$ then*

$$\begin{aligned} {}_4F_3 & \left[\begin{matrix} -n_1, -x_2 - x_1 - \beta_1, x_2 - x_1 + \beta_2 - \beta_1, 1 - n_1 - \beta_1 + \beta_0 \\ 1 - n_1 - x_1 - \beta_1, \beta_2 - \beta_1, -n_1 - \beta_1 + \beta_0 + 1 - x_1 \end{matrix}; 1 \right] \\ & = {}_4F_3 \left[\begin{matrix} -N - \tilde{x}_p - \tilde{\beta}_p, 1 - 2\tilde{N}_1^{p-1} - \tilde{n}_p - \tilde{\beta}_p + \tilde{\beta}_0, -\tilde{n}_p, N + \tilde{\beta}_{p+1} - \tilde{x}_p - \tilde{\beta}_p \\ 1 - \tilde{N}_1^p - \tilde{x}_p - \tilde{\beta}_p, \tilde{\beta}_{p+1} - \tilde{\beta}_p, -\tilde{N}_1^p - \tilde{x}_p - \tilde{\beta}_p + \tilde{\beta}_0 + 1 \end{matrix}; 1 \right] \end{aligned} \quad (4.6a)$$

and for $k = 2, 3, \dots, p-1$ we have

$$\begin{aligned} {}_4F_3 & \left[\begin{matrix} -n_k, x_{k+1} - x_k + \beta_{k+1} - \beta_k, N_1^{k-1} - \beta_0 - x_k, 1 - N_1^k + x_{k+1} \\ 1 - N_1^k - x_k - \beta_k, N_1^{k-1} + x_{k+1} + \beta_{k+1} - \beta_0, x_{k+1} - x_k - n_k + 1 \end{matrix}; 1 \right] \\ & = {}_4F_3 \left[\begin{matrix} \tilde{x}_{p+2-k} - \tilde{x}_{p+1-k} + \tilde{\beta}_{p+2-k} - \tilde{\beta}_{p+1-k}, -\tilde{n}_{p+1-k}, \\ 1 - \tilde{N}_1^{p+1-k} - \tilde{x}_{p+1-k} - \tilde{\beta}_{p+1-k}, \tilde{N}_1^{p-k} + \tilde{x}_{p+2-k} + \tilde{\beta}_{p+2-k} - \tilde{\beta}_0, \end{matrix} \right. \\ & \quad \left. 1 - \tilde{N}_1^{p+1-k} + \tilde{x}_{p+2-k}, \tilde{N}_1^{p-k} - \tilde{\beta}_0 - \tilde{x}_{p+1-k}; 1 \right] \\ & \quad \left. \tilde{x}_{p+2-k} - \tilde{x}_{p+1-k} - \tilde{n}_{p+1-k} + 1 \right]. \end{aligned} \quad (4.6b)$$

For $n \in \mathbb{N}_0^p$ let us normalize Racah polynomials as follows

$$\hat{R}_p(n; x; \beta; N) = \frac{R_p(n; x; \beta; N)}{(-N)_{|n|} (-N - \beta_0)_{|n|} \prod_{k=1}^p (\beta_{k+1} - \beta_k)_{n_k}}. \quad (4.7)$$

The main result in this subsection is the following theorem.

Theorem 4.4. *If (x, n, β) and $(\tilde{x}, \tilde{n}, \tilde{\beta})$ are related by (4.1) and if $n, \tilde{n} \in \mathbb{N}_0^p$ then*

$$\hat{R}_p(n; x; \beta; N) = \hat{R}_p(\tilde{n}; \tilde{x}; \tilde{\beta}; N). \quad (4.8)$$

Proof. Let us write $R_p(n; x; \beta; N)$ and $R_p(\tilde{n}; \tilde{x}; \tilde{\beta}; N)$ using the representation of r_{n_k} given in Lemma 4.2. Notice that according to Lemma 4.3 the ${}_4F_3$ factor in r_{n_1} is equal to the ${}_4F_3$ factor in $r_{\tilde{n}_p}$ (and therefore by Lemma 4.1 the ${}_4F_3$ factor in r_{n_p} is equal to the ${}_4F_3$ factor in $r_{\tilde{n}_1}$), and for $k = 2, 3, \dots, p-1$ the ${}_4F_3$ factor in r_{n_k} becomes the ${}_4F_3$ factor in $r_{\tilde{n}_{p+1-k}}$.

Thus if we compute the ratio $R_p(n; x; \beta; N)/R_p(\tilde{n}; \tilde{x}; \tilde{\beta}; N)$ all ${}_4F_3$'s will cancel and we get

$$\begin{aligned} \frac{R_p(n; x; \beta; N)}{R_p(\tilde{n}; \tilde{x}; \tilde{\beta}; N)} & = \\ & \times \prod_{k=1,p} \frac{(1 - N_1^k - x_k - \beta_k)_{n_k} (\beta_{k+1} - \beta_k)_{n_k} (-N_1^k - \beta_k + \beta_0 + 1 - x_k)_{n_k}}{(1 - \tilde{N}_1^k - \tilde{x}_k - \tilde{\beta}_k)_{\tilde{n}_k} (\tilde{\beta}_{k+1} - \tilde{\beta}_k)_{\tilde{n}_k} (-\tilde{N}_1^k - \tilde{\beta}_k + \tilde{\beta}_0 + 1 - \tilde{x}_k)_{\tilde{n}_k}} \\ & \times \prod_{k=2}^{p-1} \frac{(1 - N_1^k - x_k - \beta_k)_{n_k} (N_1^{k-1} + x_{k+1} + \beta_{k+1} - \beta_0)_{n_k} (x_{k+1} - x_k - n_k + 1)_{n_k}}{(1 - \tilde{N}_1^k - \tilde{x}_k - \tilde{\beta}_k)_{\tilde{n}_k} (\tilde{N}_1^{k-1} + \tilde{x}_{k+1} + \tilde{\beta}_{k+1} - \tilde{\beta}_0)_{\tilde{n}_k} (\tilde{x}_{k+1} - \tilde{x}_k - \tilde{n}_k + 1)_{\tilde{n}_k}}. \end{aligned} \quad (4.9)$$

Using (4.1)-(4.2) we can eliminate x and \tilde{x} on the right-hand side in (4.9) and we can write it as the product $T_1 T_2 T_3$, where

$$\begin{aligned} T_1 &= \prod_{k=1}^p \frac{(\tilde{N}_1^{p+1-k} + N_1^{k-1} - N)_{n_k}}{(N_1^{p+1-k} + \tilde{N}_1^{k-1} - N)_{\tilde{n}_k}} \\ T_2 &= \frac{(\beta_2 - \beta_1)_{n_1}}{(\tilde{\beta}_2 - \tilde{\beta}_1)_{\tilde{n}_1}} \prod_{k=2}^{p-1} \frac{(\beta_{k+1} - \beta_k + \tilde{n}_{p+1-k})_{n_k}}{(\tilde{\beta}_{k+1} - \tilde{\beta}_k + n_{p+1-k})_{\tilde{n}_k}} \frac{(\beta_{p+1} - \beta_p)_{n_p}}{(\tilde{\beta}_{p+1} - \tilde{\beta}_p)_{\tilde{n}_p}} \\ T_3 &= \frac{(\tilde{N}_1^p - N - \beta_0)_{n_1}}{(N_1^p - N - \tilde{\beta}_0)_{\tilde{n}_1}} \prod_{k=2}^{p-1} \frac{(N_1^{k-1} + \tilde{N}_1^{p-k} - \beta_0 - N)_{n_k}}{(\tilde{N}_1^{k-1} + N_1^{p-k} - \tilde{\beta}_0 - N)_{\tilde{n}_k}} \frac{(\tilde{N}_1^1 + N_1^{p-1} - N - \beta_0)_{n_p}}{(N_1^1 + \tilde{N}_1^{p-1} - N - \tilde{\beta}_0)_{\tilde{n}_p}}. \end{aligned}$$

Using

$$(\tilde{N}_1^{p+1-k} + N_1^{k-1} - N)_{n_k} = \frac{(-N)_{\tilde{N}_1^{p+1-k} + N_1^k}}{(-N)_{\tilde{N}_1^{p+1-k} + N_1^{k-1}}}$$

for the numerator of T_1 and an analogous formula for the denominator we see that

$$T_1 = \frac{(-N)_{N_1^p}}{(-N)_{\tilde{N}_1^p}} = \frac{(-N)_{|n|}}{(-N)_{|\tilde{n}|}}.$$

For T_2 we write

$$(\beta_{k+1} - \beta_k + \tilde{n}_{p+1-k})_{n_k} = \frac{(\beta_{k+1} - \beta_k)_{\tilde{n}_{p+1-k} + n_k}}{(\beta_{k+1} - \beta_k)_{\tilde{n}_{p+1-k}}}.$$

Using that $\beta_{k+1} - \beta_k = \tilde{\beta}_{p+2-k} - \tilde{\beta}_{p+1-k}$ and rearranging the terms one can check that

$$T_2 = \prod_{k=1}^p \frac{(\beta_{k+1} - \beta_k)_{n_k}}{(\tilde{\beta}_{k+1} - \tilde{\beta}_k)_{\tilde{n}_k}}.$$

Finally, a similar argument shows that

$$T_3 = \frac{(-N - \beta_0)_{N_1^p}}{(-N - \tilde{\beta}_0)_{\tilde{N}_1^p}} = \frac{(-N - \beta_0)_{|n|}}{(-N - \tilde{\beta}_0)_{|\tilde{n}|}}.$$

The proof now follows immediately from (4.9) and the formulas for T_1 , T_2 and T_3 . \square

4.2. The dual algebra \mathcal{A}_n . We denote by $\mathcal{D}_x^{\beta, N}$ the associative subalgebra of \mathcal{D}_x of difference operators with coefficients depending rationally on the parameters (β, N) . Clearly, the commutative algebra \mathcal{A}_x defined in Theorem 3.9 is contained in $\mathcal{D}_x^{\beta, N}$. Similarly, we denote by $\mathcal{D}_n^{\beta, N}$ the associative algebra of difference operators in the variables $n = (n_1, n_2, \dots, n_p)$ with coefficients depending rationally on n and the parameters (β, N) .

Replacing i by $p+1-k$ in (4.1c) we see that

$$n_{p+1-k} = \tilde{x}_k - \tilde{x}_{k+1} + \beta_k - \beta_{k+1}.$$

From this equation it follows that a forward shift in the variable \tilde{x}_k will correspond to a forward shift in n_{p+1-k} and a backward shift in n_{p+2-k} for $k = 2, 3, \dots, p$ and

to a forward shift in n_p when $k = 1$. Thus, in view of the duality established in Theorem 4.4, we define a function \mathbf{b} as follows

$$\mathbf{b}(N) = N \quad (4.10a)$$

$$\mathbf{b}(\beta_0) = \beta_0 \quad (4.10b)$$

$$\mathbf{b}(\beta_k) = \beta_0 - \beta_{p+2-k} - 2N + 1 \text{ for } k = 1, 2, \dots, p+1 \quad (4.10c)$$

$$\mathbf{b}(x_k) = n_1 + n_2 + \dots + n_{p+1-k} + \beta_{p+2-k} - \beta_0 + N - 1 \quad (4.10d)$$

$$\mathbf{b}(E_{x_k}) = E_{n_{p+1-k}} E_{n_{p+2-k}}^{-1}, \text{ for } k = 1, 2, \dots, p \quad (4.10e)$$

with the convention that $E_{n_{p+1}}$ is the identity operator.

Lemma 4.5. *The mapping (4.10) extends to an isomorphism $\mathbf{b} : \mathcal{D}_x^{\beta, N} \rightarrow \mathcal{D}_n^{\beta, N}$. In particular, the operators \mathfrak{L}_k^n defined by*

$$\mathfrak{L}_k^n = \mathbf{b}(\mathfrak{L}_k^x) \quad \text{for } k = 1, 2, \dots, p \quad (4.11)$$

commute with each other and therefore

$$\mathcal{A}_n = \mathbf{b}(\mathcal{A}_x) = \mathbb{R}[\mathfrak{L}_1^n, \mathfrak{L}_2^n, \dots, \mathfrak{L}_p^n] \quad (4.12)$$

is a commutative subalgebra of $\mathcal{D}_n^{\beta, N}$.

Proof. Using (4.10) one can check that

$$\mathbf{b}(E_{x_k}) \cdot \mathbf{b}(f(x)) = \mathbf{b}(f(x + e_k)) \mathbf{b}(E_{x_k}),$$

holds for every $k = 1, 2, \dots, p$, which shows that \mathbf{b} is a well defined homomorphism from $\mathcal{D}_x^{\beta, N}$ to $\mathcal{D}_n^{\beta, N}$. The fact that \mathbf{b} is one-to-one and onto follows easily. \square

For $j = 1, 2, \dots, p$ we denote

$$\begin{aligned} \kappa_j(x) &= \mathbf{b}^{-1}(\mu_j(n)) = -(x_{p+1-j} - N)(x_{p+1-j} + \beta_{p+1-j} + N) \\ &= -\lambda_{p+1-j}(x_{p+1-j}) + N(N + \beta_{p+1-j}). \end{aligned} \quad (4.13)$$

The main result in the paper is the following theorem.

Theorem 4.6. *The polynomials $\hat{R}_p(n; x; \beta; N)$ defined by equations (3.10) and (4.7) diagonalize the algebras \mathcal{A}_x and \mathcal{A}_n . More precisely, the following spectral equations hold*

$$\mathfrak{L}_j^x \hat{R}_p(n; x; \beta; N) = \mu_j(n) \hat{R}_p(n; x; \beta; N) \quad (4.14a)$$

$$\mathfrak{L}_j^n \hat{R}_p(n; x; \beta; N) = \kappa_j(x) \hat{R}_p(n; x; \beta; N), \quad (4.14b)$$

for $j = 1, 2, \dots, p$ where \mathfrak{L}_j^x, μ_j are given by (3.24) and $\mathfrak{L}_j^n, \kappa_j$ are given in (4.11), (4.13).

Remark 4.7 (Boundary conditions). Since \mathfrak{L}_j^n contains backward shift operators, and since $\hat{R}_p(n; x; \beta; N)$ is defined only for $n \in \mathbb{N}_0^p$ it is natural to ask what happens when \mathfrak{L}_j^n produces a term with a negative n_i for some i . From (4.10e) it follows that $\mathbf{b}(E_x^\nu)$ with $\nu \in \{-1, 0, 1\}^p$ will contain a negative power of $E_{n_{p+2-k}}$ in one of the following two cases:

Case 1: $\nu_k = 1, \nu_{k-1} = 0$. In this case, $\mathbf{b}(E_x^\nu)$ will contain $E_{n_{p+2-k}}^{-1}$. Notice that the coefficient of E_x^ν is C_ν which has the factor $(x_k - x_{k-1} + \beta_k - \beta_{k-1})$ (in $B_{k-1}^{0,1}$ - see formula (3.16a)). Since $\mathbf{b}(x_k - x_{k-1} + \beta_k - \beta_{k-1}) = -n_{p+2-k}$ we see that this term is 0 when $n_{p+2-k} = 0$.

Case 2: $\nu_k = 1, \nu_{k-1} = -1$. This time $\mathbf{b}(E_x^\nu)$ contains $E_{n_{p+2-k}}^{-2}$. The coefficient

C_ν has the factor $B_{k-1}^{-1,1} = (x_k - x_{k-1} + \beta_k - \beta_{k-1})(x_k - x_{k-1} + \beta_k - \beta_{k-1} + 1)$ (see (3.18b)). Since $\mathfrak{b}(B_{k-1}^{-1,1}) = n_{p+2-k}(n_{p+2-k} - 1)$ we see that this coefficient is 0 when $n_{p+2-k} = 0$ or 1.

Proof of Theorem 4.6. Equation (4.14a) follows immediately from Theorem 3.9 and the fact that $\hat{R}_p(n; x; \beta; N)$ and $R_p(n; x; \beta; N)$ differ by a factor independent of x . It remains to prove (4.14b).

Consider first the case when $N \notin \mathbb{N}_0$. Then $\hat{R}_p(n; x; \beta; N)$ are defined for all $n \in \mathbb{N}_0^p$. Let us fix n, β, N . By Lemma 4.1 for every $x \in \mathbb{R}^p$ there exist unique $(\tilde{x}, \tilde{n}, \tilde{\beta})$ such that $\mathfrak{f}(\tilde{x}, \tilde{n}, \tilde{\beta}) = (x, n, \beta)$. If $\tilde{n} \in \mathbb{N}_0^p$ then we can use Theorem 4.4 and replace $\hat{R}_p(n; x; \beta; N)$ by $\hat{R}_p(\tilde{n}; \tilde{x}; \tilde{\beta}; N)$. Equation (4.14b) follows from the fact that the operator \mathfrak{L}_j^n in the variables n with parameters β, N coincides with the operator \mathfrak{L}_j^x in the variables \tilde{x} with parameters $\tilde{\beta}, N$. Now we can think as follows: fix n, β, N and write x in terms of the dual variables \tilde{n} , i.e. we put

$$x_k = \tilde{n}_1 + \tilde{n}_2 + \cdots + \tilde{n}_{p+1-k} - \beta_k - N \text{ for } k = 1, 2, \dots, p.$$

Both sides of equation (4.14b) are polynomials in \tilde{n} of total degree at most $|n| + 1$. Since (4.14b) is true for every $\tilde{n} \in \mathbb{N}_0^p$, we conclude that it must be true for arbitrary $\tilde{n} \in \mathbb{R}^p$, or equivalently, for arbitrary $x \in \mathbb{R}^p$.

If $N \in \mathbb{N}_0$ we can obtain equation (4.14b) by a limiting procedure. In this case, we consider the polynomials $\hat{R}_p(n; x; \beta; N)$ for $n \in \mathbb{N}_0^p$ such that $|n| \leq N$ and equation (4.14b) holds when $|n| < N$. When $|n| = N$ all terms in (4.14b) have well defined limits, but the right-hand side will contain also polynomials of degree $N + 1$. Indeed, from (4.10e) it follows that $\mathfrak{b}(E_x^\nu) = E_n^{\nu'}$ where $|\nu'| = 0$ or $|\nu'| = \pm 1$, and $|\nu'| = 1$ if and only if $\nu_1 = 1$. In this case C_ν has $x_1 + \beta_1$ as a factor (coming from $B_0^{0,1}$, see (3.1b)). Since $\mathfrak{b}(x_1 + \beta_1) = |n| - N$ this term is equal to 0 when $|n| = N$. Notice that the polynomials $R_p(n; x; \beta; N)$ are well defined even when $|n| \geq N + 1$ (since all denominators in (3.9) coming from the hypergeometric series cancel). From (4.7) it follows that if $N \in \mathbb{N}_0$ and if $n, k \in \mathbb{N}_0^p$ are such that $|n| = N$, $|k| = N + 1$, then $\lim_{N' \rightarrow N} (|n| - N') \hat{R}_p(k; x; \beta; N')$ is a well defined polynomial. Thus, each term on the left-hand side of (4.14b) has a natural limit when $N \in \mathbb{N}_0$. \square

Remark 4.8. Using the isomorphism \mathfrak{b} in Lemma 4.5 we can also define involutions I_j^n on the dual algebra $\mathcal{D}_n^{\beta, N}$ by

$$I_j^n = \mathfrak{b} \circ I_j \circ \mathfrak{b}^{-1}, \quad \text{for } j = 1, 2, \dots, p.$$

Then the operators \mathfrak{L}_i^n , $i = 1, 2, \dots, p$ will be invariant under the action of the involutions I_j^n , $j = 1, 2, \dots, p$.

In the one dimensional case, this involution played a crucial role in the construction of orthogonal polynomials satisfying higher-order differential or q -difference equations, by applying the Darboux transformation to the second order difference operator \mathfrak{L}^n corresponding to the Jacobi polynomials [11] and the Askey-Wilson polynomials [13].

5. EXAMPLES OF OTHER BISPECTRAL FAMILIES OF ORTHOGONAL POLYNOMIALS

It is well known that the Racah and the Wilson polynomials are at the top of the Askey-scheme [19]. In this section we show that a change of the variables and

the parameters leads to bispectral algebras of difference operators for multivariable Wilson polynomials. We consider also in detail several limiting cases related to multivariable Hahn, Jacobi, Krawtchouk and Meixner polynomials, illustrating their bispectrality.

5.1. Wilson polynomials. If we set $\alpha = a + b - 1$, $\beta = c + d - 1$, $\gamma = a + d - 1$, $\delta = a - d$ and $x = -a - iy$ in (3.9) we obtain the Wilson polynomials

$$\begin{aligned} w_n(y; a, b, c, d) &= (a+b)_n(a+c)_n(a+d)_n \\ &\times {}_4F_3 \left[\begin{matrix} -n, n+a+b+c+d-1, a+iy, a-iy \\ a+b, a+c, a+d \end{matrix}; 1 \right]. \end{aligned} \quad (5.1)$$

If we make the change of variables

$$\beta_0 = a - b \quad (5.2a)$$

$$\beta_k = 2\mathcal{E}_2^k + 2a \quad (5.2b)$$

$$\beta_{p+1} = 2\mathcal{E}_2^p + 2a + c + d \quad (5.2c)$$

$$x_k = -\mathcal{E}_2^k - a - iy_k \quad (5.2d)$$

$$N = -\mathcal{E}_2^p - a - d, \quad (5.2e)$$

where $k = 1, 2, \dots, p$, $\mathcal{E}_2^k = \epsilon_2 + \epsilon_3 + \dots + \epsilon_k$, $\mathcal{E}_2^1 = 0$, we see that the Racah polynomials $R_p(n; x; \beta; N)$ defined by (3.10) transform into the multivariable Wilson polynomials $W_p(n; y; a, b, c, d, \epsilon)$ defined by Tratnik , see [26, formula (2.1) on page 2066]. When

$$\operatorname{Re}(a, b, c, d, \epsilon_2, \epsilon_3, \dots, \epsilon_p) > 0$$

these polynomials are orthogonal in \mathbb{R}^p with respect to the measure

$$\begin{aligned} d\rho'_{a,b,c,d,\epsilon}(y) &= \Gamma(a + iy_1)\Gamma(a - iy_1)\Gamma(b + iy_1)\Gamma(b - iy_1) \times \\ &\prod_{k=1}^{p-1} \frac{\Gamma(\epsilon_{k+1} + iy_{k+1} + iy_k)\Gamma(\epsilon_{k+1} + iy_{k+1} - iy_k)\Gamma(\epsilon_{k+1} - iy_{k+1} + iy_k)\Gamma(\epsilon_{k+1} - iy_{k+1} - iy_k)}{\Gamma(2iy_k)\Gamma(-2iy_k)} \\ &\times \frac{\Gamma(c + iy_p)\Gamma(c - iy_p)\Gamma(d + iy_p)\Gamma(d - iy_p)}{\Gamma(2iy_p)\Gamma(-2iy_p)} dy. \end{aligned}$$

From (5.2) it is clear that the shift E_{x_k} corresponds to the shift operator $E_{y_k}^i$ in the variables y defined by $E_{y_k}^i f(y) = f(y + ie_k)$. Thus the bispectral properties established in Theorem 4.6 lead to bispectral commutative algebras \mathcal{A}_y and \mathcal{A}_n for $W_p(n; y; a, b, c, d, \epsilon)$ after the change of variables (5.2).

5.2. Hahn polynomials. Following Tratnik [27] we can obtain the multivariable Hahn polynomials of Karlin and McGregor [18] as follows. We introduce new parameters $\{\gamma_1, \gamma_2, \dots, \gamma_{p+1}\}$ by

$$\beta_k = \beta_0 + \gamma_1^k + k, \quad \text{for } k = 1, 2, \dots, p+1, \quad (5.3a)$$

where $\gamma_1^k = \gamma_1 + \gamma_2 + \dots + \gamma_k$, $\gamma_1^0 = 0$ and new variables y_1, y_2, \dots, y_p by

$$y_k = x_k - x_{k-1} \quad \text{for } k = 1, 2, \dots, p, \quad (5.3b)$$

where $x_0 = 0$. Then

$$\beta_{k+1} - \beta_k = \gamma_{k+1} + 1 \quad \text{for } k = 1, 2, \dots, p.$$

Up to a factor independent of x , the weight $\rho(x)$ in (3.8) can be written as

$$\rho'(x) = \prod_{k=0}^p \frac{(\beta_{k+1} - \beta_k)_{x_{k+1}-x_k} (\beta_{k+1})_{x_{k+1}+x_k}}{(x_{k+1} - x_k)! (\beta_k + 1)_{x_{k+1}+x_k}} \prod_{k=1}^p \frac{\beta_k + 2x_k}{\beta_0}.$$

Thus, if we let $\beta_0 \rightarrow \infty$ we see that $\rho'(x)$ approaches

$$\rho_{\gamma, N}(y) = \prod_{k=1}^p \frac{(\gamma_k + 1)_{y_k}}{y_k!} \frac{(\gamma_{p+1} + 1)_{N-|y|}}{(N - |y|)!} = \prod_{k=1}^{p+1} \frac{(\gamma_k + 1)_{y_k}}{y_k!}, \quad (5.4)$$

with the convention that $y_{p+1} = N - |y|$. For $a, b, N \in \mathbb{R}$ and $n \in \mathbb{N}_0$ we denote by $h_n(x; a; b; N)$ the one dimensional Hahn polynomials

$$h_n(x; a; b; N) = (a+1)_n (-N)_n {}_3F_2 \left[\begin{matrix} -n, n+a+b+1, -x \\ a+1, -N \end{matrix}; 1 \right].$$

Then when $\beta_0 \rightarrow \infty$ the polynomials $\hat{R}_p(n; x; \beta; N)$ defined by (3.10) and (4.7) become the multivariable Hahn polynomials

$$\begin{aligned} H_p(n; y; \gamma; N) &= \frac{(-1)^{|n|}}{(-N)_{|n|} \prod_{k=1}^p (\gamma_{k+1} + 1)_{n_k}} \\ &\times \prod_{k=1}^p h_{n_k}(Y_1^k - N_1^{k-1}; 2N_1^{k-1} + \gamma_1^k + k - 1; \gamma_{k+1}; Y_1^{k+1} - N_1^{k-1}), \end{aligned} \quad (5.5)$$

orthogonal for $N \in \mathbb{N}_0$ with respect to the weight $\rho_{\gamma, N}(y)$ given in (5.4) on the set $V = \{y \in \mathbb{N}_0^p : |y| \leq N\}$. Here we use again the convention $Y_1^k = y_1 + y_2 + \dots + y_k$. Formula (5.5) above coincides (up to a normalization factor) with the formulas given by Karlin and McGregor, see [18, p. 277]. Since the weight is invariant under an arbitrary permutation of the labels $(1, 2, \dots, p+1)$, one can easily generate different orthogonal families, corresponding to different ways of applying the Gram-Schmidt process in the vector space of polynomials of total degree k , for $k = 0, 1, 2, \dots, N$.

Let us now focus on the difference operators $\{\mathcal{L}_j^x, \mathcal{L}_j^n\}_{j=1}^p$ and the eigenvalues $\{\mu_j(n), \kappa_j(x)\}_{j=1}^p$ in Theorem 4.6 when we change the parameters and the variables according to (5.3) and then let $\beta_0 \rightarrow \infty$. First we change the parameters using (5.3a). We can write

$$\mathcal{L}_j^x = \sum_{0 \neq \nu \in \{0, \pm 1\}^j} C_\nu^{(j)}(\beta_0, \gamma, N)(E_x^\nu - 1),$$

where $C_\nu^{(j)}(\beta_0, \gamma, N)$, $B_i^{\nu_i, \nu_{i+1}}(\beta_0, \gamma, N)$, etc. denote the coefficients and their components in the new parameters $\beta_0, \gamma_1, \dots, \gamma_{p+1}, N$. From the explicit formulas (3.1)-(3.2) (see also (3.14)-(3.18)) it is easy to see that

$$\lim_{\beta_0 \rightarrow \infty} \frac{B_i^{\nu_i, \nu_{i+1}}(\beta_0, \gamma, N)}{\beta_0^2} = \begin{cases} \frac{1}{2} & \text{if } \nu_i = \nu_{i+1} = 0 \\ 1 & \text{if } \nu_i = \nu_{i+1} = \pm 1 \end{cases}$$

$$\lim_{\beta_0 \rightarrow \infty} \frac{B_i^{\nu_i, \nu_{i+1}}(\beta_0, \gamma, N)}{\beta_0} \text{ is finite when } \nu_i \neq \nu_{i+1},$$

and

$$\lim_{\beta_0 \rightarrow \infty} \frac{b_i^{\nu_i}(\beta_0, \gamma, N)}{\beta_0^2} = 1.$$

This combined with (3.3) shows that if for $i = 0, 1, \dots, j$ we have at least three pairs (ν_i, ν_{i+1}) such that $\nu_i \neq \nu_{i+1}$ then $\lim_{\beta_0 \rightarrow \infty} C_\nu^{(j)}(\beta_0, \gamma, N) = 0$. Thus we need to consider only $\nu \neq 0$ of the form: $\nu_l = \nu_{l+1} = \dots = \nu_k = \pm 1$ for some $1 \leq l \leq k \leq j$ and all other coordinates of ν are equal to 0. Notice also that the operator $E_{x_l} E_{x_{l+1}} \cdots E_{x_k}$ corresponds to the operator $E_{y_l} E_{y_{k+1}}^{-1}$ in the variables y_1, y_2, \dots, y_p . Here we adopt the convention that $E_{y_{p+1}}$ is the identity operator. If we put $m = k + 1$ then after the change of variables and parameters (5.3) the operator \mathfrak{L}_j^x reduces to the operator

$$\tilde{\mathfrak{L}}_j^y = \sum_{1 \leq l \neq m \leq j+1} (y_l + \gamma_l + 1) y_m (E_{y_l} E_{y_m}^{-1} - 1). \quad (5.6)$$

This operator plays a crucial role in the work [18]. Another useful representation of $\tilde{\mathfrak{L}}_j^y$ is given below

$$\begin{aligned} \tilde{\mathfrak{L}}_j^y = & - \sum_{1 \leq l \neq m \leq j+1} (y_l + \gamma_l + 1) y_m \Delta_{y_l} \nabla_{y_m} + \sum_{m=1}^{j+1} y_m (Y_1^{j+1} - y_m) \Delta_{y_m} \nabla_{y_m} \\ & + \sum_{m=1}^{j+1} (\gamma_m + 1) (Y_1^{j+1} - y_m) \Delta_{y_m} - \sum_{m=1}^{j+1} (\gamma_1^{j+1} - \gamma_m + j) y_m \nabla_{y_m}, \end{aligned} \quad (5.7)$$

with the convention that $\Delta_{y_{p+1}}$ and $\nabla_{y_{p+1}}$ are zero operators, or equivalently the sums are up to p when $j = p$.

Next we consider the difference operators in n . Let us denote by $L_\nu^{(j)}$ the coefficients of the operator $\mathfrak{L}_j^n = \mathfrak{L}_j^n(\beta_0, \gamma, N)$ defined by (4.11) in the variables (β_0, γ, N) , i.e. we can write

$$\mathfrak{L}_j^n(\beta_0, \gamma, N) = \sum_{0 \neq \nu \in \{0, \pm 1\}^j} L_\nu^{(j)} (\mathfrak{b}(E_x^\nu) - 1),$$

where

$$L_\nu^{(j)} = \mathfrak{b}(C_\nu) = 2^{j-|\nu^+|-|\nu^-|} \frac{\prod_{k=0}^j \tilde{B}_k^{\nu_k, \nu_{k+1}}}{\prod_{k=1}^j \tilde{b}_k^{\nu_k}}. \quad (5.8)$$

On the right-hand side of the last formula, we have $\tilde{B}_k^{\nu_k, \nu_{k+1}}$ and $\tilde{b}_k^{\nu_k}$ which denote $\mathfrak{b}(B_k^{\nu_k, \nu_{k+1}})$ and $\mathfrak{b}(b_k^{\nu_k})$, respectively in the parameters β_0, γ, N .

Lemma 5.1. *The product*

$$\prod_{k=1}^j \frac{\tilde{B}_k^{\nu_k, \nu_{k+1}}}{\tilde{b}_k^{\nu_k}}$$

is independent of β_0 and N , i.e. all terms in (5.8) except \tilde{B}_0^{0, ν_1} will not be affected by the limit $\beta_0 \rightarrow \infty$. Moreover,

$$\lim_{\beta_0 \rightarrow \infty} \frac{\tilde{B}_0^{0, \nu_1}}{\beta_0} = N + \text{terms involving } n \text{ and } \gamma. \quad (5.9)$$

Proof. Notice first that $\mathfrak{b}(\beta_k)$ and $\mathfrak{b}(x_k)$ are independent of β_0 after the change of variables (5.3). Using also the fact that $\mathfrak{b}(x_{i+1} - x_i)$, $\mathfrak{b}(\beta_{i+1} - \beta_i)$, $\mathfrak{b}(x_{i+1} + x_i + \beta_{i+1})$, $\mathfrak{b}(x_{i+1} + x_i + \beta_i)$ are independent of N for $i \geq 1$, we obtain the first assertion of the Lemma. Equation (5.9) can be easily checked by considering the three possible cases $\nu_1 = 0, 1, -1$. \square

Based on the above Lemma we define

$$\tilde{\mathcal{L}}_j^n = \lim_{\beta_0 \rightarrow \infty} \frac{1}{\beta_0} \mathcal{L}_j^n(\beta_0, \gamma, N), \text{ for } j = 1, 2, \dots, p, \quad (5.10)$$

where in the right-hand side the operator is written in terms of β_0 and γ , using (5.3). Then as a corollary of Theorem 4.6 we obtain the following bispectral property of the multivariable Hahn polynomials.

Theorem 5.2. *The polynomials $H_p(n; y; \gamma; N)$ defined by (5.5) satisfy the following spectral equations*

$$\tilde{\mathcal{L}}_j^y H_p(n; y; \gamma; N) = \tilde{\mu}_j(n) H_p(n; y; \gamma; N) \quad (5.11a)$$

$$\tilde{\mathcal{L}}_j^n H_p(n; y; \gamma; N) = \tilde{\kappa}_j(y) H_p(n; y; \gamma; N), \quad (5.11b)$$

for $j = 1, 2, \dots, p$ where $\tilde{\mathcal{L}}_j^y$ and $\tilde{\mathcal{L}}_j^n$ are given by (5.6) and (5.10), respectively, and

$$\tilde{\mu}_j(n) = -(n_1 + n_2 + \dots + n_j)(n_1 + n_2 + \dots + n_j + j + \gamma_1 + \dots + \gamma_{j+1})$$

$$\tilde{\kappa}_j(y) = N - y_1 - y_2 - \dots - y_{p+1-j}.$$

Remark 5.3. It is interesting to notice that while the operators $\tilde{\mathcal{L}}_j^y$ are significantly simpler than \mathcal{L}_j^x (for instance, $\tilde{\mathcal{L}}_p^y$ has $p(p+1)$ terms with coefficients which are quadratic polynomials of y , while \mathcal{L}_p^x has 3^p terms with coefficients which are rational functions of x) the operators $\tilde{\mathcal{L}}_j^n$ and \mathcal{L}_j^n have exactly the same number of components, and they both have rational coefficients of n . This phenomenon appears even when we take one more limit and we consider the Jacobi polynomials. This is the content of the next subsection.

5.3. Jacobi polynomials. From the multivariable Hahn polynomials we can obtain multivariable Jacobi polynomials as follows. We introduce new variables z_1, z_2, \dots, z_p by

$$y_k = Nz_k, \quad \text{for } k = 1, \dots, p. \quad (5.12)$$

Using the conventions in the previous subsection, we define $z_{p+1} = 1 - |z|$ and $Z_1^k = z_1 + z_2 + \dots + z_k$. Applying the substitution (5.12), we see that the k -th term h_{n_k} in (5.5) satisfies

$$\begin{aligned} \lim_{N \rightarrow \infty} \frac{h_{n_k}}{N^{n_k}} &= (-Z_1^{k+1})^{n_k} (2N_1^{k-1} + \gamma_1^k + k)_{n_k} \\ &\times {}_2F_1 \left[\begin{matrix} -n_k, n_k + 2N_1^{k-1} + \gamma_1^{k+1} + k \\ 2N_1^{k-1} + \gamma_1^k + k \end{matrix}; \frac{Z_1^k}{Z_1^{k+1}} \right]. \end{aligned} \quad (5.13)$$

Let us denote by $P_n(x; \alpha, \beta)$ the usual Jacobi polynomials

$$P_n(x; \alpha, \beta) = \frac{(\alpha + 1)_n}{n!} {}_2F_1 \left[\begin{matrix} -n, n + \alpha + \beta + 1 \\ \alpha + 1 \end{matrix}; \frac{1-x}{2} \right].$$

Using (5.13) we see that if we make the substitution (5.12) in (5.5) and let $N \rightarrow \infty$ the multivariable Hahn polynomials become the multivariable Jacobi polynomials

$$J_p(n; z; \gamma) = (-1)^{|n|} \prod_{k=1}^p \frac{n_k!(Z_1^{k+1})^{n_k}}{(\gamma_{k+1} + 1)_{n_k}} P_{n_k} \left(1 - 2 \frac{Z_1^k}{Z_1^{k+1}}; 2N_1^{k-1} + \gamma_1^k + k - 1, \gamma_{k+1} \right). \quad (5.14)$$

These polynomials are orthogonal on the simplex

$$T^p = \{z \in \mathbb{R}^p : z_i \geq 0 \text{ for } i = 1, 2, \dots, p \text{ and } |z| = z_1 + z_2 + \dots + z_p \leq 1\}$$

with respect to the weight

$$\rho_\gamma(z) = \prod_{k=1}^p z_k^{\gamma_k} (1 - |z|)^{\gamma_{p+1}}. \quad (5.15)$$

Notice that

$$\lim_{N \rightarrow \infty} N\Delta_{y_k} = \lim_{N \rightarrow \infty} N\nabla_{y_k} = \frac{\partial}{\partial z_k}.$$

Thus, using (5.7) we see that if we make the change of variables (5.12) and let $N \rightarrow \infty$, the operators $\tilde{\mathcal{L}}_j^y$ become the differential operators

$$\begin{aligned} \hat{\mathcal{L}}_j^z &= -2 \sum_{1 \leq l < m \leq \min(j+1, p)} z_l z_m \frac{\partial^2}{\partial z_l \partial z_m} + \sum_{m=1}^{\min(j+1, p)} z_m (Z_1^{j+1} - z_m) \frac{\partial^2}{\partial z_m^2} \\ &\quad + \sum_{m=1}^{\min(j+1, p)} \left((\gamma_m + 1) Z_1^{j+1} - (\gamma_1^{j+1} + j + 1) z_m \right) \frac{\partial}{\partial z_m}. \end{aligned} \quad (5.16)$$

When $j = p$ we obtain the operator

$$\begin{aligned} \hat{\mathcal{L}}_p^z &= -2 \sum_{1 \leq l < m \leq p} z_l z_m \frac{\partial^2}{\partial z_l \partial z_m} + \sum_{m=1}^p z_m (1 - z_m) \frac{\partial^2}{\partial z_m^2} \\ &\quad + \sum_{m=1}^p \left((\gamma_m + 1) - (\gamma_1^{p+1} + p + 1) z_m \right) \frac{\partial}{\partial z_m}. \end{aligned}$$

This operator is independent of the way we apply the Gram-Schmidt process in each vector space of polynomials of total degree k , for $k = 0, 1, 2, \dots$, i.e. if we construct orthogonal polynomials on T^p with respect to weight (5.15) they will be eigenfunctions of $\hat{\mathcal{L}}_p^z$. The operator $\hat{\mathcal{L}}_p^z$ was derived at the beginning of the last century in the monograph [1] in the case $p = 2$, from the differential equations satisfied by the Lauricella functions. The operator for general p can also be obtained by similar techniques, see [17]. Other proofs use Dunkl's differential-difference operators, see [9] for details.

We define difference operators in n by

$$\hat{\mathcal{L}}_j^n = \lim_{N \rightarrow \infty} \frac{1}{N} \tilde{\mathcal{L}}_j^N. \quad (5.17)$$

By Lemma 5.1 these operators will have the same supports as the operators $\tilde{\mathcal{L}}_j^N$. More precisely, using (5.8) we can write

$$\hat{\mathcal{L}}_j^n(\gamma) = \sum_{0 \neq \nu \in \{0, \pm 1\}^j} \hat{L}_\nu^{(j)}(\mathfrak{b}(E_x^\nu) - 1),$$

where

$$\hat{L}_\nu^{(j)} = 2^{j - |\nu^+| - |\nu^-|} \prod_{k=1}^j \frac{\tilde{B}_k^{\nu_k, \nu_{k+1}}}{\tilde{b}_k^{\nu_k}}.$$

Combining the above formulas with Theorem 5.2 we obtain the bispectrality of the multivariable Jacobi polynomials on the simplex T^p with weight function given by (5.15).

Theorem 5.4. *The polynomials $J_p(n; z; \gamma)$ defined by (5.14) satisfy the following spectral equations*

$$\hat{\mathfrak{L}}_j^z J_p(n; z; \gamma) = \hat{\mu}_j(n) J_p(n; z; \gamma) \quad (5.18a)$$

$$\hat{\mathfrak{L}}_j^n J_p(n; z; \gamma) = \hat{\kappa}_j(z) J_p(n; y; \gamma), \quad (5.18b)$$

for $j = 1, 2, \dots, p$ where $\hat{\mathfrak{L}}_j^z$ and $\hat{\mathfrak{L}}_j^n$ are given by (5.16) and (5.17), respectively, and

$$\hat{\mu}_j(n) = -(n_1 + n_2 + \dots + n_j)(n_1 + n_2 + \dots + n_j + j + \gamma_1 + \dots + \gamma_{j+1})$$

$$\hat{\kappa}_j(z) = 1 - z_1 - z_2 - \dots - z_{p+1-j}.$$

5.4. Krawtchouk and Meixner polynomials. Multivariable Krawtchouk and Meixner polynomials can be obtained from the multivariable Hahn polynomials [25, 27], which is one way to construct the bispectral operators. On the other hand, they possess interesting duality properties, which combined with the explicit form of the admissible difference operators [16], allows us to write simple formulas for the commutative algebras on both sides. Below we sketch the main ingredients of this construction, which parallels the theory developed for the Racah polynomials in the first sections. For interesting applications of Krawtchouk and Meixner polynomials to multi-dimensional linear growth birth and death processes see [22].

For $n \in \mathbb{N}_0^p$ and parameters $\mathbf{p} = (\mathfrak{p}_1, \mathfrak{p}_2, \dots, \mathfrak{p}_p)$ and N , we define multivariable Krawtchouk polynomials by

$$K_p(n; x; \mathbf{p}; N) = \frac{1}{(-N)_{|n|}} \prod_{j=1}^p k_{n_j} \left(x_j; \frac{\mathfrak{p}_j}{1 - \mathfrak{P}_1^{j-1}}; N - |n| + N_1^j - X_1^{j-1} \right), \quad (5.19)$$

where as before, for $y \in \mathbb{R}^p$ we set $Y_1^j = y_1 + y_2 + \dots + y_j$, with the convention $Y_1^0 = 0$, and k_{n_j} are the one dimensional Krawtchouk polynomials

$$k_n(x; \mathbf{p}; N) = (-N)_n {}_2F_1 \left[\begin{matrix} -n, -x \\ -N \end{matrix} ; \frac{1}{\mathfrak{p}} \right] \text{ for } n \in \mathbb{N}_0 \text{ and } x, \mathbf{p}, N \in \mathbb{R}.$$

The multivariable polynomials defined by (5.19) are eigenfunctions of the operator

$$\mathcal{L}_p(x; \mathbf{p}; N) = \sum_{1 < i \neq j < p} \mathfrak{p}_i x_j \Delta_{x_i} \nabla_{x_j} + \sum_{i=1}^p [\mathfrak{p}_i (x_i - N) \Delta_{x_i} + (1 - \mathfrak{p}_i) x_i \nabla_{x_i}] \quad (5.20)$$

with eigenvalue $\mu_p(n) = |n|$. When $N \in \mathbb{N}_0$ and $|\mathbf{p}| < 1$ these polynomials are orthogonal on $\{x \in \mathbb{N}_0^p : |x| \leq N\}$ with respect to the weight

$$\rho(x) = \frac{1}{(N - |x|)!} \prod_{k=1}^p \frac{1}{x_k!} \left(\frac{\mathfrak{p}_k}{1 - |\mathbf{p}|} \right)^{x_k}.$$

Remark 5.5. If we put

$$\mathfrak{p}_k = \frac{c_k}{|c| - 1} \text{ and } N = -s \quad (5.21)$$

in (5.19) we obtain the multivariable Meixner polynomials $M(n; x; c; s)$. When c_k and s are positive with $|c| < 1$, these polynomials are orthogonal on \mathbb{N}_0^p with respect to the weight

$$\rho_{c,s}(x) = (s)_{|x|} \prod_{k=1}^p \frac{(c_k)^{x_k}}{x_k!}.$$

Thus all difference equations derived for the Krawtchouk polynomials will be true for the Meixner polynomials after changing the parameters according to (5.21).

Let us now introduce dual variables and parameters for the Krawtchouk polynomials by

$$x_k = \tilde{n}_{p+1-k} \quad (5.22a)$$

$$n_k = \tilde{x}_{p+1-k} \quad (5.22b)$$

$$\mathfrak{p}_k = \frac{\tilde{\mathfrak{p}}_{p+1-k}(1 - |\tilde{\mathfrak{p}}|)}{(1 - \tilde{\mathfrak{P}}_1^{p+1-k})(1 - \tilde{\mathfrak{P}}_1^{p-k})}, \quad (5.22c)$$

for $k = 1, 2, \dots, p$. Then one can deduce the following analog of Lemma 4.1 and Theorem 4.4. In this case the duality follows immediately and we do not need any transformations for the ${}_2F_1$'s.

Theorem 5.6. *The mapping $\mathfrak{f} : (\tilde{x}, \tilde{n}, \tilde{\mathfrak{p}}) \rightarrow (x, n, \mathfrak{p})$ given by (5.22) is a bijection $\mathbb{R}^{3p} \rightarrow \mathbb{R}^{3p}$ such that $\mathfrak{f}^{-1} = \mathfrak{f}$. Moreover, if $n, \tilde{n} \in \mathbb{N}_0^p$ then the polynomials defined by (5.19) satisfy the following duality relation*

$$K_p(n; x; \mathfrak{p}; N) = K_p(\tilde{n}; \tilde{x}; \tilde{\mathfrak{p}}; N). \quad (5.23)$$

Notice next that if we consider in (5.19) the product of the terms for $k = j$ to p , then we obtain (up to an unessential factor) the multivariable Krawtchouk polynomial

$$K_{p+1-j} \left(n_j, n_{j+1}, \dots, n_p; x_j, x_{j+1}, \dots, x_p; \right. \\ \left. \frac{\mathfrak{p}_j}{1 - \mathfrak{P}_1^{j-1}}, \frac{\mathfrak{p}_{j+1}}{1 - \mathfrak{P}_1^{j-1}}, \dots, \frac{\mathfrak{p}_p}{1 - \mathfrak{P}_1^{j-1}}; N - X_1^{j-1} \right).$$

Since the product of the terms for $k = 1, 2, \dots, j-1$ is independent of the variables x_j, x_{j+1}, \dots, x_p we define for $j = 1, 2, \dots, p$ the operators

$$\mathfrak{L}_j^x = \mathcal{L}_{p+1-j} \left(x_j, x_{j+1}, \dots, x_p; \frac{\mathfrak{p}_j}{1 - \mathfrak{P}_1^{j-1}}, \frac{\mathfrak{p}_{j+1}}{1 - \mathfrak{P}_1^{j-1}}, \dots, \frac{\mathfrak{p}_p}{1 - \mathfrak{P}_1^{j-1}}; N - X_1^{j-1} \right). \quad (5.24)$$

Thus we obtain the commutative algebra

$$\mathcal{A}_x = \mathbb{R}[\mathfrak{L}_1^x, \mathfrak{L}_2^x, \dots, \mathfrak{L}_p^x].$$

If we denote by $\mathcal{D}_x^{\mathfrak{p}, N}$ and $\mathcal{D}_n^{\mathfrak{p}, N}$ the associative algebras of operators in the variables x and n , respectively, with coefficients depending rationally on the parameters \mathfrak{p}, N , we define an isomorphism $\mathfrak{b} : \mathcal{D}_x^{\mathfrak{p}, N} \rightarrow \mathcal{D}_n^{\mathfrak{p}, N}$ by

$$\mathfrak{b}(x_k) = n_{p+1-k}, \quad \mathfrak{b}(E_{x_k}) = E_{n_{p+1-k}} \quad k = 1, 2, \dots, p \quad (5.25a)$$

$$\mathfrak{b}(\mathfrak{p}_k) = \frac{\mathfrak{p}_{p+1-k}(1 - |\mathfrak{p}|)}{(1 - \mathfrak{P}_1^{p+1-k})(1 - \mathfrak{P}_1^{p-k})} \quad k = 1, 2, \dots, p \quad (5.25b)$$

$$\mathfrak{b}(N) = N. \quad (5.25c)$$

This leads to the dual commutative algebra

$$\mathcal{A}_n = \mathbb{R}[\mathfrak{L}_1^n, \mathfrak{L}_2^n, \dots, \mathfrak{L}_p^n], \text{ where } \mathfrak{L}_j^n = \mathfrak{b}(\mathfrak{L}_j^x).$$

Theorem 5.7. *The polynomials $K_p(n; x; \mathfrak{p}; N)$ defined by equations (5.19) diagonalize the algebras \mathcal{A}_x and \mathcal{A}_n . More precisely, the following spectral equations hold*

$$\mathfrak{L}_j^x K_p(n; x; \mathfrak{p}; N) = (n_j + n_{j+1} + \cdots + n_p) K_p(n; x; \mathfrak{p}; N) \quad (5.26a)$$

$$\mathfrak{L}_j^n K_p(n; x; \mathfrak{p}; N) = (x_1 + x_2 + \cdots + x_{p+1-j}) K_p(n; x; \mathfrak{p}; N), \quad (5.26b)$$

for $j = 1, 2, \dots, p$.

APPENDIX A. EXPLICIT FORMULAS IN DIMENSION TWO

A.1. Racah polynomials. From (3.10) and (4.7) we have

$$\begin{aligned} \hat{R}_2(n; x; \beta; N) &= \frac{r_{n_1}(\beta_1 - \beta_0 - 1, \beta_2 - \beta_1 - 1, -x_2 - 1, \beta_1 + x_2; x_1)}{(-N)_{n_1+n_2}(-N - \beta_0)_{n_1+n_2}(\beta_2 - \beta_1)_{n_1}(\beta_3 - \beta_2)_{n_2}} \\ &\times r_{n_2}(2n_1 + \beta_2 - \beta_0 - 1, \beta_3 - \beta_2 - 1, n_1 - N - 1, n_1 + \beta_2 + N; -n_1 + x_2). \end{aligned}$$

The operator \mathfrak{L}_2^x can be written as follows

$$\mathfrak{L}_2^x = \sum_{0 \neq \nu \in \{-1, 0, 1\}^2} C_\nu^{(2)} (E_{x_1}^{\nu_1} E_{x_2}^{\nu_2} - 1)$$

where the coefficients C_ν are given by

$$\begin{aligned} C_{(1,1)}^{(2)} &= \frac{(x_1 + \beta_1)(x_1 + \beta_1 - \beta_0)(x_2 + x_1 + \beta_2)(x_2 + x_1 + \beta_2 + 1)(N - x_2)(N + x_2 + \beta_3)}{(2x_1 + \beta_1)(2x_1 + \beta_1 + 1)(2x_2 + \beta_2)(2x_2 + \beta_2 + 1)} \\ C_{(1,0)}^{(2)} &= \frac{(x_1 + \beta_1)(x_1 + \beta_1 - \beta_0)(x_2 - x_1)(x_2 + x_1 + \beta_2)(2\lambda_2 + 2\lambda_3 + (\beta_2 + 1)(\beta_3 - 1))}{(2x_1 + \beta_1)(2x_1 + \beta_1 + 1)(2x_2 + \beta_2 + 1)(2x_2 + \beta_2 - 1)} \\ C_{(0,1)}^{(2)} &= \frac{(2\lambda_1 + (\beta_0 + 1)(\beta_1 - 1))(x_2 + x_1 + \beta_2)(x_2 - x_1 + \beta_2 - \beta_1)(N - x_2)(N + x_2 + \beta_3)}{(2x_1 + \beta_1 + 1)(2x_1 + \beta_1 - 1)(2x_2 + \beta_2)(2x_2 + \beta_2 + 1)} \\ C_{(-1,1)}^{(2)} &= I_1(C_{(1,1)}^{(2)}), \quad C_{(1,-1)}^{(2)} = I_2(C_{(1,1)}^{(2)}), \quad C_{(-1,-1)}^{(2)} = I_1(I_2(C_{(1,1)}^{(2)})) \\ C_{(-1,0)}^{(2)} &= I_1(C_{(1,0)}^{(2)}), \quad C_{(0,-1)}^{(2)} = I_2(C_{(0,1)}^{(2)}) \end{aligned}$$

where $\lambda_1 = x_1(x_1 + \beta_1)$, $\lambda_2 = x_2(x_2 + \beta_2)$, $\lambda_3 = N(N + \beta_3)$.

Similarly we have

$$\mathfrak{L}_1^x = C_1^{(1)}(E_{x_1} - 1) + C_{-1}^{(1)}(E_{x_1}^{-1} - 1),$$

where

$$C_1^{(1)} = \frac{(x_1 + \beta_1 - \beta_0)(x_1 + \beta_1)(x_2 + x_1 + \beta_2)(x_2 - x_1)}{(2x_1 + \beta_1)(2x_1 + \beta_1 + 1)} \text{ and } C_{-1}^{(1)} = I_1(C_1^{(1)}).$$

The corresponding eigenvalues are

$$\begin{aligned} \mu_2(n) &= -(n_1 + n_2)(n_1 + n_2 - 1 + \beta_3 - \beta_0) \\ \mu_1(n) &= -n_1(n_1 - 1 + \beta_2 - \beta_0). \end{aligned}$$

Next we write the explicit formulas for the dual operators \mathfrak{L}_2^n and \mathfrak{L}_1^n . First we write

$$\mathfrak{L}_2^n = \sum_{\nu \in S} D_\nu^{(2)} (E_{n_1}^{\nu_1} E_{n_2}^{\nu_2} - 1),$$

where $S = \{(1, 0), (0, 1), (-1, 2), (1, -1), (-1, 1), (0, -1), (1, -2), (-1, 0)\}$ and the coefficients are given by

$$\begin{aligned}
D_{(1,0)}^{(2)} &= \frac{(n_1 + n_2 - N)(n_1 + n_2 - N - \beta_0)(2n_1 + n_2 + \beta_3 - \beta_0 - 1)(2n_1 + n_2 + \beta_3 - \beta_0)}{(2n_1 + 2n_2 + \beta_3 - \beta_0 - 1)(2n_1 + 2n_2 + \beta_3 - \beta_0)} \\
&\quad \times \frac{(-n_1 - \beta_2 + \beta_0 + 1)(n_1 + \beta_2 - \beta_1)}{(2n_1 + \beta_2 - \beta_0 - 1)(2n_1 + \beta_2 - \beta_0)} \\
D_{(0,1)}^{(2)} &= \frac{(n_1 + n_2 - N)(n_1 + n_2 - N - \beta_0)(-n_2 + \beta_2 - \beta_3)(2n_1 + n_2 + \beta_3 - \beta_0 - 1)T_2(n)}{(2n_1 + 2n_2 + \beta_3 - \beta_0 - 1)(2n_1 + 2n_2 + \beta_3 - \beta_0)(2n_1 + \beta_2 - \beta_0)(2n_1 + \beta_2 - \beta_0 - 2)} \\
D_{(-1,2)}^{(2)} &= \frac{(n_1 + n_2 - N)(n_1 + n_2 - N - \beta_0)(-n_2 + \beta_2 - \beta_3)}{(2n_1 + 2n_2 + \beta_3 - \beta_0 - 1)(2n_1 + 2n_2 + \beta_3 - \beta_0)} \\
&\quad \times \frac{(-n_2 + \beta_2 - \beta_3 - 1)n_1(-n_1 + \beta_0 - \beta_1 + 1)}{(2n_1 + \beta_2 - \beta_0 - 1)(2n_1 + \beta_2 - \beta_0 - 2)} \\
D_{(1,-1)}^{(2)} &= \frac{T_0(n)(2n_1 + n_2 + \beta_3 - \beta_0 - 1)n_2(n_1 + \beta_2 - \beta_0 - 1)(n_1 + \beta_2 - \beta_1)}{(2n_1 + 2n_2 + \beta_3 - \beta_0)(2n_1 + 2n_2 + \beta_3 - \beta_0 - 2)(2n_1 + \beta_2 - \beta_0 - 1)(2n_1 + \beta_2 - \beta_0)} \\
D_{(-1,1)}^{(2)} &= \frac{T_0(n)(-n_2 + \beta_2 - \beta_3)(2n_1 + n_2 + \beta_2 - \beta_0 - 1)n_1(-n_1 + \beta_0 - \beta_1 + 1)}{(2n_1 + 2n_2 + \beta_3 - \beta_0)(2n_1 + 2n_2 + \beta_3 - \beta_0 - 2)(2n_1 + \beta_2 - \beta_0 - 1)(2n_1 + \beta_2 - \beta_0 - 2)} \\
D_{(0,-1)}^{(2)} &= -\frac{(n_1 + n_2 + \beta_3 - \beta_0 + N - 1)(n_1 + n_2 + \beta_3 + N - 1)(2n_1 + n_2 + \beta_2 - \beta_0 - 1)n_2T_2(n)}{(2n_1 + 2n_2 + \beta_3 - \beta_0 - 1)(2n_1 + 2n_2 + \beta_3 - \beta_0 - 2)(2n_1 + \beta_2 - \beta_0)(2n_1 + \beta_2 - \beta_0 - 2)} \\
D_{(1,-2)}^{(2)} &= \frac{(n_1 + n_2 + \beta_3 - \beta_0 + N - 1)(n_1 + n_2 + \beta_3 + N - 1)n_2(n_2 - 1)(-n_1 - \beta_2 + \beta_0 + 1)(n_1 + \beta_2 - \beta_1)}{(2n_1 + 2n_2 + \beta_3 - \beta_0 - 1)(2n_1 + 2n_2 + \beta_3 - \beta_0 - 2)(2n_1 + \beta_2 - \beta_0 - 1)(2n_1 + \beta_2 - \beta_0)} \\
D_{(-1,0)}^{(2)} &= \frac{(n_1 + n_2 + \beta_3 - \beta_0 + N - 1)(n_1 + n_2 + \beta_3 + N - 1)(2n_1 + n_2 + \beta_2 - \beta_0 - 1)}{(2n_1 + 2n_2 + \beta_3 - \beta_0 - 1)(2n_1 + 2n_2 + \beta_3 - \beta_0 - 2)} \\
&\quad \times \frac{(2n_1 + n_2 + \beta_2 - \beta_0 - 2)n_1(-n_1 + \beta_0 - \beta_1 + 1)}{(2n_1 + \beta_2 - \beta_0 - 1)(2n_1 + \beta_2 - \beta_0 - 2)},
\end{aligned}$$

where we have set

$$\begin{aligned}
T_0(n) &= \mathbf{b}(2B_0^{0,0}) = -2\mu_2(n) + (\beta_0 + 1)(\beta_0 - \beta_3) - 2N(N + \beta_3) \\
T_2(n) &= \mathbf{b}(2B_2^{0,0}) = -2\mu_1(n) + (\beta_0 - \beta_1)(\beta_0 - \beta_2 + 2).
\end{aligned}$$

For \mathfrak{L}_1^n we have

$$\mathfrak{L}_1^n = D_{(0,1)}^{(1)}(E_{n_2} - 1) + D_{(0,-1)}^{(1)}(E_{n_2}^{-1} - 1),$$

where

$$D_{(0,1)}^{(1)} = \frac{(n_1 + n_2 - N - \beta_0)(n_1 + n_2 - N)(2n_1 + n_2 + \beta_3 - \beta_0 - 1)(-n_2 + \beta_2 - \beta_3)}{(2n_1 + 2n_2 + \beta_3 - \beta_0 - 1)(2n_1 + 2n_2 + \beta_3 - \beta_0)}$$

and

$$D_{(0,-1)}^{(1)} = -\frac{(n_1 + n_2 + \beta_3 + N - 1)(n_1 + n_2 + \beta_3 - \beta_0 + N - 1)n_2(2n_1 + n_2 + \beta_2 - \beta_0 - 1)}{(2n_1 + 2n_2 + \beta_3 - \beta_0 - 1)(2n_1 + 2n_2 + \beta_3 - \beta_0 - 2)}.$$

Finally the eigenvalues are

$$\begin{aligned}\kappa_2(x) &= -\lambda_1(x_1) + N(N + \beta_1) \\ \kappa_1(x) &= -\lambda_2(x_2) + N(N + \beta_2).\end{aligned}$$

A.2. Jacobi polynomials. When $p = 2$ formula (5.14) gives

$$\begin{aligned}J_2(n; z; \gamma) &= \frac{(-1)^{n_1+n_2} n_1! n_2! (z_1 + z_2)^{n_1}}{(\gamma_2 + 1)_{n_1} (\gamma_3 + 1)_{n_2}} \\ &\quad \times P_{n_1} \left(1 - \frac{2z_1}{z_1 + z_2}; \gamma_1, \gamma_2 \right) P_{n_2}(1 - 2(z_1 + z_2); 2n_1 + \gamma_1 + \gamma_2 + 1, \gamma_3).\end{aligned}$$

On the z side we have

$$\begin{aligned}\hat{\mathfrak{L}}_2^z &= -2z_1 z_2 \frac{\partial^2}{\partial z_1 \partial z_2} + z_1(1 - z_1) \frac{\partial^2}{\partial z_1^2} + z_2(1 - z_2) \frac{\partial^2}{\partial z_2^2} \\ &\quad + ((\gamma_1 + 1) - (\gamma_1 + \gamma_2 + \gamma_3 + 3)z_1) \frac{\partial}{\partial z_1} + ((\gamma_2 + 1) - (\gamma_1 + \gamma_2 + \gamma_3 + 3)z_2) \frac{\partial}{\partial z_2} \\ \hat{\mathfrak{L}}_1^z &= -2z_1 z_2 \frac{\partial^2}{\partial z_1 \partial z_2} + z_1 z_2 \frac{\partial^2}{\partial z_1^2} + z_1 z_2 \frac{\partial^2}{\partial z_2^2} \\ &\quad + ((\gamma_1 + 1)z_2 - (\gamma_2 + 1)z_1) \frac{\partial}{\partial z_1} + ((\gamma_2 + 1)z_1 - (\gamma_1 + 1)z_2) \frac{\partial}{\partial z_2}.\end{aligned}$$

Next we write the explicit formulas for the dual operators $\hat{\mathfrak{L}}_2^n$ and $\hat{\mathfrak{L}}_1^n$. We can write

$$\hat{\mathfrak{L}}_2^n = \sum_{\nu \in S} D_\nu^{(2)} (E_{n_1}^{\nu_1} E_{n_2}^{\nu_2} - 1),$$

where $S = \{(1, 0), (0, 1), (-1, 2), (1, -1), (-1, 1), (0, -1), (1, -2), (-1, 0)\}$ and the coefficients are given by

$$\begin{aligned}D_{(1,0)}^{(2)} &= -\frac{(2n_1 + n_2 + \gamma_1 + \gamma_2 + \gamma_3 + 2)(2n_1 + n_2 + \gamma_1 + \gamma_2 + \gamma_3 + 3)}{(2n_1 + 2n_2 + \gamma_1 + \gamma_2 + \gamma_3 + 2)(2n_1 + 2n_2 + \gamma_1 + \gamma_2 + \gamma_3 + 3)} \\ &\quad \times \frac{(n_1 + \gamma_1 + \gamma_2 + 1)(n_1 + \gamma_2 + 1)}{(2n_1 + \gamma_1 + \gamma_2 + 1)(2n_1 + \gamma_1 + \gamma_2 + 2)} \\ D_{(0,1)}^{(2)} &= -\frac{(n_2 + \gamma_3 + 1)(2n_1 + n_2 + \gamma_1 + \gamma_2 + \gamma_3 + 2)}{(2n_1 + 2n_2 + \gamma_1 + \gamma_2 + \gamma_3 + 2)(2n_1 + 2n_2 + \gamma_1 + \gamma_2 + \gamma_3 + 3)} \\ &\quad \times \frac{(2n_1(n_1 + 1 + \gamma_1 + \gamma_2) + (\gamma_1 + 1)(\gamma_1 + \gamma_2))}{(2n_1 + \gamma_1 + \gamma_2 + 2)(2n_1 + \gamma_1 + \gamma_2)} \\ D_{(-1,2)}^{(2)} &= -\frac{(n_2 + \gamma_3 + 1)(n_2 + \gamma_3 + 2)}{(2n_1 + 2n_2 + \gamma_1 + \gamma_2 + \gamma_3 + 2)(2n_1 + 2n_2 + \gamma_1 + \gamma_2 + \gamma_3 + 3)} \\ &\quad \times \frac{n_1(n_1 + \gamma_1)}{(2n_1 + \gamma_1 + \gamma_2 + 1)(2n_1 + \gamma_1 + \gamma_2)} \\ D_{(1,-1)}^{(2)} &= -\frac{2(2n_1 + n_2 + \gamma_1 + \gamma_2 + \gamma_3 + 2)n_2}{(2n_1 + 2n_2 + \gamma_1 + \gamma_2 + \gamma_3 + 3)(2n_1 + 2n_2 + \gamma_1 + \gamma_2 + \gamma_3 + 1)} \\ &\quad \times \frac{(n_1 + 1 + \gamma_1 + \gamma_2)(n_1 + \gamma_2 + 1)}{(2n_1 + \gamma_1 + \gamma_2 + 1)(2n_1 + \gamma_1 + \gamma_2 + 2)}\end{aligned}$$

$$\begin{aligned}
D_{(-1,1)}^{(2)} &= -\frac{2(n_2 + \gamma_3 + 1)(2n_1 + n_2 + \gamma_1 + \gamma_2 + 1)}{(2n_1 + 2n_2 + \gamma_1 + \gamma_2 + \gamma_3 + 3)(2n_1 + 2n_2 + \gamma_1 + \gamma_2 + \gamma_3 + 1)} \\
&\quad \times \frac{n_1(n_1 + \gamma_1)}{(2n_1 + \gamma_1 + \gamma_2 + 1)(2n_1 + \gamma_1 + \gamma_2)} \\
D_{(0,-1)}^{(2)} &= -\frac{(2n_1 + n_2 + \gamma_1 + \gamma_2 + 1)n_2}{(2n_1 + 2n_2 + \gamma_1 + \gamma_2 + \gamma_3 + 2)(2n_1 + 2n_2 + \gamma_1 + \gamma_2 + \gamma_3 + 1)} \\
&\quad \times \frac{(2n_1(n_1 + 1 + \gamma_1 + \gamma_2) + (\gamma_1 + 1)(\gamma_1 + \gamma_2))}{(2n_1 + \gamma_1 + \gamma_2 + 2)(2n_1 + \gamma_1 + \gamma_2)} \\
D_{(1,-2)}^{(2)} &= -\frac{n_2(n_2 - 1)}{(2n_1 + 2n_2 + \gamma_1 + \gamma_2 + \gamma_3 + 2)(2n_1 + 2n_2 + \gamma_1 + \gamma_2 + \gamma_3 + 1)} \\
&\quad \times \frac{(n_1 + \gamma_1 + \gamma_2 + 1)(n_1 + \gamma_2 + 1)}{(2n_1 + \gamma_1 + \gamma_2 + 1)(2n_1 + \gamma_1 + \gamma_2 + 2)} \\
D_{(-1,0)}^{(2)} &= -\frac{(2n_1 + n_2 + \gamma_1 + \gamma_2 + 1)(2n_1 + n_2 + \gamma_1 + \gamma_2)}{(2n_1 + 2n_2 + \gamma_1 + \gamma_2 + \gamma_3 + 2)(2n_1 + 2n_2 + \gamma_1 + \gamma_2 + \gamma_3 + 1)} \\
&\quad \times \frac{n_1(n_1 + \gamma_1)}{(2n_1 + \gamma_1 + \gamma_2 + 1)(2n_1 + \gamma_1 + \gamma_2)}.
\end{aligned}$$

The operator $\hat{\mathcal{L}}_1^n$ can be written as

$$\hat{\mathcal{L}}_1^n = D_{(0,1)}^{(1)}(E_{n_2} - 1) + D_{(0,-1)}^{(1)}(E_{n_2}^{-1} - 1),$$

where

$$D_{(0,1)}^{(1)} = -\frac{(2n_1 + n_2 + \gamma_1 + \gamma_2 + \gamma_3 + 2)(n_2 + \gamma_3 + 1)}{(2n_1 + 2n_2 + \gamma_1 + \gamma_2 + \gamma_3 + 2)(2n_1 + 2n_2 + \gamma_1 + \gamma_2 + \gamma_3 + 3)}$$

and

$$D_{(0,-1)}^{(1)} = -\frac{n_2(2n_1 + n_2 + \gamma_1 + \gamma_2 + 1)}{(2n_1 + 2n_2 + \gamma_1 + \gamma_2 + \gamma_3 + 2)(2n_1 + 2n_2 + \gamma_1 + \gamma_2 + \gamma_3 + 1)}.$$

A.3. Krawtchouk polynomials. When $p = 2$ equation (5.19) gives

$$K_2(n; x; \mathbf{p}; N) = \frac{1}{(-N)_{n_1+n_2}} k_{n_1}(x_1; \mathbf{p}_1; N - n_2) k_{n_2} \left(x_2; \frac{\mathbf{p}_2}{1 - \mathbf{p}_1}; N - x_1 \right)$$

and the following formulas hold for the generators of \mathcal{A}_x

$$\begin{aligned}
\mathcal{L}_1^x &= \mathbf{p}_1 x_2 \Delta_{x_1} \nabla_{x_2} + \mathbf{p}_2 x_1 \nabla_{x_1} \Delta_{x_2} + \mathbf{p}_1(x_1 - N) \Delta_{x_1} + (1 - \mathbf{p}_1)x_1 \nabla_{x_1} \\
&\quad + \mathbf{p}_2(x_2 - N) \Delta_{x_2} + (1 - \mathbf{p}_2)x_2 \nabla_{x_2} \\
\mathcal{L}_2^x &= \frac{\mathbf{p}_2}{1 - \mathbf{p}_1}(x_1 + x_2 - N) \Delta_{x_2} + \frac{1 - \mathbf{p}_1 - \mathbf{p}_2}{1 - \mathbf{p}_1} x_2 \nabla_{x_2}.
\end{aligned}$$

For \mathcal{A}_n we use (5.25) and we obtain

$$\begin{aligned}\mathfrak{L}_1^n &= \frac{\mathfrak{p}_2}{1-\mathfrak{p}_1} n_1 \Delta_{n_2} \nabla_{n_1} + \frac{\mathfrak{p}_1(1-\mathfrak{p}_1-\mathfrak{p}_2)}{1-\mathfrak{p}_1} n_2 \nabla_{n_2} \Delta_{n_1} \\ &\quad + \frac{\mathfrak{p}_2}{1-\mathfrak{p}_1} (n_2 - N) \Delta_{n_2} + \frac{1-\mathfrak{p}_1-\mathfrak{p}_2}{1-\mathfrak{p}_1} n_2 \nabla_{n_2} \\ &\quad + \frac{\mathfrak{p}_1(1-\mathfrak{p}_1-\mathfrak{p}_2)}{1-\mathfrak{p}_1} (n_1 - N) \Delta_{n_1} + \left(1 - \mathfrak{p}_1 + \frac{\mathfrak{p}_1\mathfrak{p}_2}{1-\mathfrak{p}_1}\right) n_1 \nabla_{n_1} \\ \mathfrak{L}_2^n &= \mathfrak{p}_1(n_1 + n_2 - N) \Delta_{n_1} + (1 - \mathfrak{p}_1) n_1 \nabla_{n_1}.\end{aligned}$$

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REFERENCES

- [1] P. Appell and J. Kampé de Fériet, *Fonctions hypergéométriques et hypersphériques - polynômes d'Hermite*, Gauthier-Villars et Cie, Paris, 1926.
- [2] R. Askey and J. Wilson, *Some basic hypergeometric orthogonal polynomials that generalize Jacobi polynomials*, Mem. Amer. Math. Soc. 54 (1985), no. 319, 55 pp.
- [3] W. N. Bailey, *Generalized hypergeometric series*, Cambridge Tracts in Mathematics and Mathematical Physics 32 (1935).
- [4] B. Bakalov, E. Horozov and M. Yakimov, *Highest weight modules over the $W_{1+\infty}$ algebra and the bispectral problem*, Duke Math. J. 93 (1998), no. 1, 41–72.
- [5] S. Bochner, *Über Sturm-Liouville'sche Polynomsysteme*, Math. Z. 29 (1929), no. 1, 730–736.
- [6] J. M. Brunat, C. Krattenthaler, A. Lascoux and A. Montes, *Some composition determinants*, Linear Algebra Appl. 416 (2006), no. 2-3, 355–364.
- [7] I. Cherednik, *Double affine Hecke algebras and Macdonald's conjectures*, Ann. of Math. (2) 141 (1995), no. 1, 191–216.
- [8] J. J. Duistermaat and F. A. Grünbaum, *Differential equations in the spectral parameter*, Comm. Math. Phys. 103 (1986), no. 2, 177–240.
- [9] C. F. Dunkl and Y. Xu, *Orthogonal polynomials of several variables*, Encyclopedia of Mathematics and its Applications, 81, Cambridge University Press, Cambridge, 2001.
- [10] F. A. Grünbaum, *Time-band limiting and the bispectral problem*, Comm. Pure Appl. Math. 47 (1994), no. 3, 307–328.
- [11] F. A. Grünbaum and M. Yakimov, *Discrete bispectral Darboux transformations from Jacobi operators*, Pacific J. Math. 204 (2002), no. 2, 395–431.
- [12] L. Haine and P. Iliev, *Commutative rings of difference operators and an adelic flag manifold*, Internat. Math. Res. Notices 2000 (2000), no. 6, 281–323.
- [13] L. Haine and P. Iliev, *Askey-Wilson type functions with bound states*, Ramanujan J. 11 (2006), no. 3, 285–329 (arXiv:math/0203136).
- [14] J. Harnad and A. Kasman (Eds.), *The Bispectral Problem (Montréal)*, CRM Proc. Lecture Notes, Vol. 14, AMS, Providence, 1998.
- [15] G. J. Heckman and E. M. Opdam, *Root systems and hypergeometric functions I*, Compositio Math. 64 (1987), no. 3, 329–352.
- [16] P. Iliev and Y. Xu, *Discrete orthogonal polynomials and difference equations of several variables*, Adv. Math. 212 (2007), no. 1, 1–36 (arXiv:math/0508039).
- [17] E. G. Kalnins and W. Miller, Jr., *Orthogonal polynomials on n -spheres: Gegenbauer, Jacobi and Heun*, In: *Topics in polynomials of one and several variables and their applications*, pp. 299–322, World Sci. Publ., River Edge, NJ, 1993.
- [18] S. Karlin and J. McGregor, *Linear growth models with many types and multidimensional Hahn polynomials*, in: *Theory and application of special functions*, pp. 261–288, ed. R. A. Askey, Math. Res. Center, Univ. Wisconsin, Publ. No. 35, Academic Press, New York, 1975.
- [19] R. Koekoek and R. F. Swarttouw, *The Askey-scheme of hypergeometric orthogonal polynomials and its q -analogue*, Report No. 98–17, Delft Univ. Tech. (1998).

- [20] T. Koornwinder, Askey-Wilson polynomials for root systems of type BC , in: *Hypergeometric functions on domains of positivity, Jack polynomials, and applications* (Tampa, FL, 1991), pp. 189–204, Contemp. Math., 138, Amer. Math. Soc., Providence, RI, 1992.
- [21] I. G. Macdonald, *Symmetric functions and Hall polynomials*, 2nd ed., Oxford University Press, 1995.
- [22] P. R. Milch, *A multi-dimensional linear growth birth and death process*, Ann. Math. Statist. 39 (1968), no. 3, 727–754.
- [23] A. F. Nikiforov, S. K. Suslov and V. B. Uvarov, *Classical orthogonal polynomials of a discrete variable*, Springer Series in Computational Physics, Springer-Verlag, Berlin, 1991.
- [24] S. Sahi, *Nonsymmetric Koornwinder polynomials and duality*, Ann. of Math. (2) 150 (1999), no. 1, 267–282.
- [25] M. V. Tratnik, *Multivariable Meixner, Krawtchouk, and Meixner-Pollaczek polynomials*, J. Math. Phys. 30 (1989), no. 12, 2740–2749.
- [26] M. V. Tratnik, *Some multivariable orthogonal polynomials of the Askey tableau-continuous families*, J. Math. Phys. 32 (1991), no. 8, 2065–2073.
- [27] M. V. Tratnik, *Some multivariable orthogonal polynomials of the Askey tableau-discrete families*, J. Math. Phys. 32 (1991), no. 9, 2337–2342.
- [28] J. F. van Diejen, *Self-dual Koornwinder-Macdonald polynomials*, Invent. Math. 126 (1996), no. 2, 319–339.
- [29] G. Wilson, *Collisions of Calogero-Moser particles and an adelic Grassmannian*, (with an appendix by I. G. Macdonald), Invent. Math. 133 (1998), no. 1, 1–41.
- [30] Y. Xu, *On discrete orthogonal polynomials of several variables*, Adv. in Appl. Math. 33 (2004), no. 3, 615–632.
- [31] J. P. Zubelli and F. Magri, *Differential equations in the spectral parameter, Darboux transformations and a hierarchy of master symmetries for KdV*, Comm. Math. Phys. 141 (1991), no. 2, 329–351.

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